Fall 09

(1) f bounded on $\mathbb{R}^n$, $M_\varepsilon(x) = \sup_{|y-x|<\varepsilon} f(y)$

(a) Show $M(x) := \lim_{\varepsilon\to 0} M_\varepsilon(x)$ exists for all x

(b) Show $M(x)$ is u.s.c., namely $\limsup_{y\to x} M(y) \le M(x)$

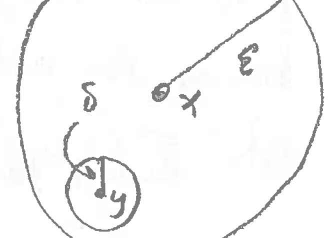
(a) See that, for $\varepsilon_1 < \varepsilon_2$,

$$M_{\varepsilon_1}(x) = \sup_{|y-x|<\varepsilon_1} f(y) = \sup_{y\in B_{\varepsilon_1}(x)} f(y) \le \sup_{y\in B_{\varepsilon_2}(x)} f(y) = M_{\varepsilon_2}(x)$$

since $B_{\varepsilon_1}(x) \subseteq B_{\varepsilon_2}(x)$; hence $M_\varepsilon(x)$ is nonincreasing as $\varepsilon \to 0$, but $M_\varepsilon(x) = \sup_{|y-x|<\varepsilon} f(y) \ge f(x)$, hence $M_\varepsilon(x)$ is bdd below since $f(x)$ is bdd. Thus $\lim_{\varepsilon\to 0} M_\varepsilon(x)$ is convergent

(b) See that $M(x) = \lim_{\varepsilon\to 0} \sup_{|y-x|<\varepsilon} f(y) = \limsup_{y\to x} f(y)$

Fix $\varepsilon > 0$, and consider $B_\varepsilon(x)$: $\longrightarrow$



Clearly, $\sup_{y\in B_\varepsilon(x)} f(y) \ge \sup_{t\in B_\delta(y)} f(t)$ since $B_\delta(y) \subseteq B_\varepsilon(x)$,

hence $\sup_{y\in B_\varepsilon(x)} f(y) \ge \sup_{t\in B_\delta(y)} f(t) \ge \lim_{\delta\to 0}\left(\sup_{t\in B_\delta(y)} f(t)\right)$

$$\Longrightarrow \sup_{y\in B_\varepsilon(x)} f(y) \ge \sup_{y\in B_\varepsilon(x)}\left(\lim_{\delta\to 0}\left(\sup_{t\in B_\delta(y)} f(t)\right)\right)$$

Now let $\varepsilon\to 0$, hence:

$$\lim_{\varepsilon\to 0}\left(\sup_{y\in B_\varepsilon(x)} f(y)\right) \ge \lim_{\varepsilon\to 0}\left(\sup_{y\in B_\varepsilon(x)}\left(\lim_{\delta\to 0}\left(\sup_{t\in B_\delta(y)} f(t)\right)\right)\right)$$

$$\limsup_{y\to x} f(y) \qquad\qquad \limsup_{y\to x}\left(\lim_{\delta\to 0}\left(\sup_{t\in B_\delta(y)} f(t)\right)\right) = \limsup_{y\to x}\left(\limsup_{t\to y} f(t)\right)$$

$$\Rightarrow \limsup_{y\to x} f(y) \ge \limsup_{y\to x}\left(\limsup_{t\to y} f(t)\right)$$

$$\Rightarrow \underline{M(x) \ge \limsup_{y\to x}(M(y))}$$

(2.) m Leb. meas on $[0,1]$; $f \in L^1(m)$, $F(x) = \int_0^x f(t)\,dt$.

Suppose φ is Lipschitz

Show : $\exists g \in L^1(m)$ s.t. $\varphi(F(x)) = \int_0^x g(t)\,dt$

- Claim : φ Lipschitz $\Rightarrow$ φ absolutely continuous :

See that $\sum_{k=1}^{n} |\varphi(x_k) - \varphi(y_k)| \leq M \sum_{k=1}^{n} |x_k - y_k|$, hence let $\delta = \frac{\varepsilon}{M}$.

Hence $\sum_{k=1}^{n} |x_k - y_k| < \frac{\varepsilon}{M} \Rightarrow M \sum_{k=1}^{\infty} |x_k - y_k| < \varepsilon \Rightarrow \sum_{k=1}^{n} |\varphi(x_k) - \varphi(y_k)|$

$\leq M \sum_{k=1}^{n} |x_k - y_k| < \varepsilon$,

here φ is absolutely continuous

- Claim : $\varphi \circ F$ is absolutely continuous :

By construction, F is absolutely cont. Choose $\varepsilon > 0$;

φ is abs. cont, hence $\exists \delta > 0$ s.t. $\sum_{k=1}^{n} |x_k - y_k| < \delta \Rightarrow \sum_{k=1}^{n} |\varphi(x_k) - \varphi(y_k)| < \varepsilon$,

but on the other hand, F is abs. cont, hence given $\delta > 0$, $\exists \delta' > 0$

s.t. $\sum (x_k - y_k) < \delta' \Rightarrow \sum |F(x_k) - F(y_k)| < \delta$

$\Rightarrow \sum |\varphi(F(x_k)) - \varphi(F(y_k))| < \varepsilon$

Hence $\varphi \circ F$ is absolutely continuous, i.e. $\exists g \in L^1$ s.t.

$\varphi \circ F(x) = \int_0^x g(t)\,dt$.

Recall : FTC Lebesgue :

TFAE ① F abs cont.

② $\exists g \in L^1$ s.t. $F(x) - F(a) = \int_a^x g(t)\,dt$

③ $E \subseteq \mathbb{R}$ has finite Lebesgue measure; show that
$$f(x) = \int_{\mathbb{R}} \chi_E(y)\chi_E(y-x)\,dy$$

Since we have Lebesgue measure, we know that $m(E) = m(E+x) = m(E+p)$, hence
$$\int \chi_E(y)dy < \infty, \quad \int \chi_{E+x}(y)dy < \infty, \quad \int \chi_{E+p}(y)\,dy < \infty,$$

Since χ_E is integrable, we have, for each $\varepsilon > 0$, an int. step function ϕ s.t. $\int |\chi_E(y) - \phi(y)| < \varepsilon$.

Since m is Lebesgue, $\phi(y) = \sum_{i=1}^{k} a_i \chi_{E_i}$ where E_i are finite unions of open intervals. Since $\chi_E = 0$ or 1, $a_i = 1$ $\forall i$.

Now, let $\varepsilon > 0$ and then:
$$|f(x) - f(p)| = \left| \int_{\mathbb{R}} \chi_E(y)\chi_E(y-x)dy - \int_{\mathbb{R}} \chi_E(y)\chi_E(y+p)\,dy \right|$$
$$\leq \int_{\mathbb{R}} |\chi_E(y)|\,|\chi_E(y-x) - \chi_E(y+p)|\,dy$$
$$\leq \mu(E) \int_{\mathbb{R}} |\chi_{E+x}(y) - \chi_{E+y}(y)|\,dy. \quad (*)$$

Now, $\int |\chi_E(y) - \phi(y)| < \varepsilon$, hence $\int |\chi_E(y-x) - \phi(y-x)| < \varepsilon$ and $\int |\chi_E(y-p) - \phi(y-p)| < \varepsilon$ by substitution (and integrability).

Let $\phi_p = \phi(y-p)$ and $\phi_x = \phi(y-x)$; now we have:
$$(*) = \mu(E) \int_{\mathbb{R}} |\chi_{E+x} - \phi_x + \phi_x - \phi_p + \phi_p - \chi_{E-p}|\,dy$$
$$\leq \mu(E) \int_{\mathbb{R}} (|\chi_{E+x} - \phi_x| + |\chi_{E+p} - \phi_p| + |\phi_x - \phi_p|)\,dy$$
$$< \mu(E)\left(\varepsilon + \varepsilon + \int_{\mathbb{R}} |\phi_x - \phi_p|\,dy\right)$$

Now, recall that $\phi(y) = \sum_{i=1}^{k} \chi_{E_i}$ where $E_i = \bigcup_{j=1}^{n} (a_j, b_j)$

(we'll use the same n and let some intervals be empty)

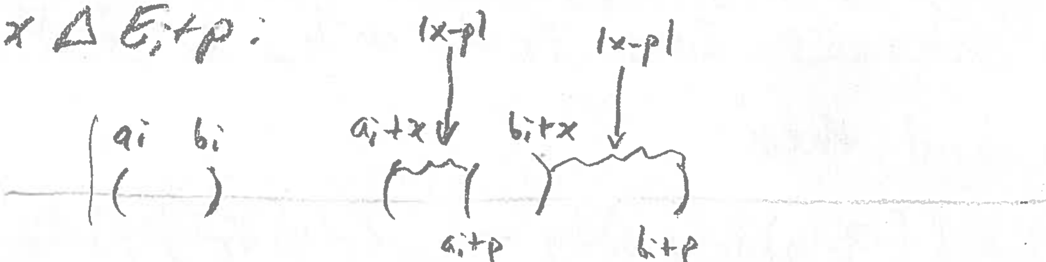
Then: $\int |\phi_x - \phi_p| \, dy = \int |\phi(y-x) - \phi(y-p)| \, dy$

$$= \int \left| \sum_{i=1}^{k} \chi_{E_i}(y-x) - \sum_{i=1}^{k} \chi_{E_i}(y-p) \right| dy$$

$$= \int \left| \sum_{i=1}^{k} \chi_{E_i+x}(y) - \sum_{i=1}^{k} \chi_{E_i+p}(y) \right| dy$$

$$= \int \left| \sum_{i=1}^{k} \chi_{E_i+x \Delta E_i+p}(y) \right| dy = \sum_{i=1}^{k} \int \chi_{E_i+x \Delta E_i+p}(y) \, dy$$

Recall that E_i is a union of intervals, hence consider $E_i + x \Delta E_i + p$:



$$m\left[\left(\bigcup_{j=1}^{n} (a_j, b_j) + x\right) \Delta \left(\bigcup_{j=1}^{n} (a_j, b_j) + p\right)\right] = m\left[\bigcup_{j=1}^{n} (a_j+x, b_j+x) \Delta \bigcup_{j=1}^{n} (a_j+p, b_j+p)\right]$$

$$\leq 2n|x-p|$$

Hence $m(E_i + x \Delta E_i + p) \leq 2n|x-p|$,

hence $\sum_{i=1}^{k} \int \chi_{E_i+x \Delta E_i+p}(y) \, dy = \sum_{i=1}^{k} \mu(E_i + x \Delta E_i + p) \leq 2kn|x-p|$

Hence:

$$|f(x) - f(p)| < \mu(E)\left(\varepsilon + \varepsilon + \int_{\mathbb{R}} |\phi_x - \phi_p| \, dy\right)$$

$$\leq \mu(E)(\varepsilon + \varepsilon + 2kn|x-p|)$$

Hence if $|x-p| < \frac{1}{kn}\left(\frac{\varepsilon}{2\mu(E)} - \varepsilon\right)$, then $|f(x) - f(p)| < \varepsilon$,

hence f is continuous.

(4.) m Lebesgue measure on $\mathbb{R}$, $f_n, f \in L^1(m)$, $\|f_n - f\|_{L^1} \le \frac{C}{n^2}$ for all $n \ge 1$. Show $f_n \to f$ a.e.

Recall: $f_n(x) \to f(x)$ if for $\varepsilon > 0$, $\exists N > 0$ s.t. $n \ge N \Rightarrow |f_n(x) - f(x)| < \varepsilon$,
hence $f_n(x) \not\to f(x)$ if $\exists \varepsilon > 0$ s.t. $\forall N > 0$, $\exists n \ge N$ s.t. $|f_n(x) - f(x)| > \varepsilon$

Hence we'll show that:

$A = \{x : \exists \varepsilon > 0$ s.t. for all $N > 0$, $\exists n \ge N \Rightarrow |f_n(x) - f(x)| > \varepsilon\}$ is meas 0.

Consider the sets $A_{\varepsilon,N} = \{x : |f_n(x) - f(x)| \ge \varepsilon$ for some $n \ge N\}$

Then $A = \bigcup_{\varepsilon > 0} \{x :$ for all $N > 0$, $|f_n(x) - f(x)| > \varepsilon$ for some $n \ge N\}$

$$= \bigcup_{\varepsilon > 0} \bigcap_{N=1}^{\infty} \{x : |f_n(x) - f(x)| > \varepsilon \text{ for some } n \ge N\}$$

$$= \bigcup_{\varepsilon > 0} \bigcap_{N=1}^{\infty} A_{\varepsilon,N} \quad ; \text{ let } \bigcap_{N=1}^{\infty} A_{\varepsilon,N} = A_\varepsilon$$

→ The A_ε are an increasing sequence as $\varepsilon \to 0$:

$$\varepsilon_1 > \varepsilon_2 \Rightarrow A_{\varepsilon_1} = \{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon_1 > \varepsilon_2$$
$$\subseteq \{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon_2\} = A_{\varepsilon_2}$$

Hence apply continuity from below:

$$m\left(\bigcup_{\varepsilon > 0} A_\varepsilon\right) = \lim_{\varepsilon \to 0} m(A_\varepsilon) = \lim_{\varepsilon \to 0} m\left(\bigcap_{N=1}^{\infty} A_{\varepsilon,N}\right)$$

→ The $A_{\varepsilon,N}$ are a decreasing sequence as $N \to \infty$

$$N_1 > N_2 \Rightarrow A_{\varepsilon,N_1} = \{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon \text{ some } n \ge N_1 > N_2\}$$
$$\supseteq \{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon \text{ some } n \ge N_2\} = A_{\varepsilon,N_2}$$

(since if $|f_n(x) - f(x)| > \varepsilon$ for $n \ge N_2$, then $|f_n(x) - f(x)| > \varepsilon$ for $n \ge N_1 > N_2$)

and see that $m(A_{\varepsilon,1}) = m(\{x : |f_n(x) - f(x)| > \varepsilon$ for some $n \ge 1\})$,

so:
$$\varepsilon\, m(A_{\varepsilon,1}) = \int \varepsilon \chi_{A_{\varepsilon,1}}\, dm = \int_{A_{\varepsilon,1}} \varepsilon\, dm < \int_{A_{\varepsilon,1}} |f_n - f|\, dm \le \int_{\mathbb{R}} |f_n - f|\, dm \le \frac{C}{n^2}$$

$\Rightarrow m(A_{\varepsilon,1}) < \infty$ so now we may apply continuity from above:

$$\lim_{\varepsilon\to 0} m\left(\bigcap_{N=1}^{\infty} A_{\varepsilon,N}\right) = \lim_{\varepsilon\to 0}\lim_{N\to\infty} m(A_{\varepsilon,N})$$

Now:

$$\varepsilon\, m(A_{N,\varepsilon}) = \int_{A_{N,\varepsilon}} \varepsilon\, dm < \int_{A_{N,\varepsilon}} |f_n - f|\, dm < \int_{\mathbb{R}} |f_n - f|\, dm \le \frac{C}{n^2}$$

$\Rightarrow$ for $n \geq N$, hence $n^2 \geq N^2$, hence $\frac{1}{n^2} \le \frac{1}{N^2}$

hence $m(A_{N,\varepsilon}) \le \frac{C}{N^2\varepsilon}$

$$\Rightarrow m(A) = \lim_{\varepsilon\to 0}\lim_{N\to\infty} m(A_{\varepsilon,N}) \le \lim_{\varepsilon\to 0}\lim_{N\to\infty} \frac{C}{N^2\varepsilon} = \lim_{\varepsilon\to 0} 0 = 0$$

$\Rightarrow m(A) = 0 \Rightarrow$ <u>$f_n \to f$ a.e.</u>