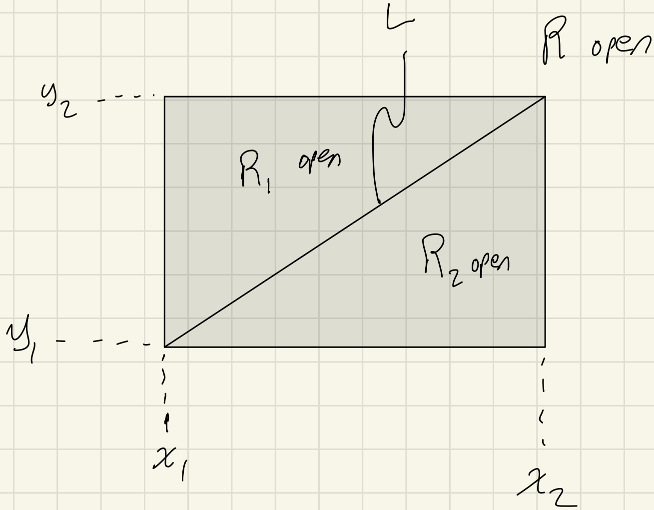
08 Fa. 1

(1) Let μ, ν be finite Borel measures on $\mathbb{R}^2$ such that $\mu(B) = \nu(B)$ for every open triangular region B in the plane. Show that $\mu(E) = \nu(E)$ for all Borel sets E. [Note added later: This is a modified version of the problem actually asked, which was inappropriately difficult, with balls in place of triangles.]

$R = L \sqcup R_1 \sqcup R_2$

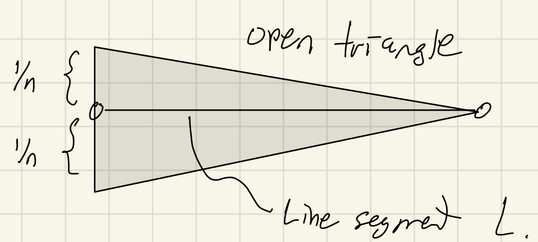


$$\mu(R) = \mu(L) + \mu(R_1) + \mu(R_2)$$
$$= \mu(L) + \nu(R_1) + \nu(R_2)$$

So, if $\mu(L) = \nu(L)$ then $\mu(R) = \nu(R)$. And we'd be done since $\mathcal{B}(\mathbb{R}^2)$ generated by rectangles.

So, it's left to show agreement on a line segment.

Let $\{L_n\}$ be a collection of open triangles as here. We see that $L = \bigcap_1^\infty L_n$.



Since, μ, ν finite we can use continuity from above to say that $\mu(L) = \lim_n \mu(L_n) = \lim_n \nu(L_n) = \nu(L)$.

$\square$

08 F.2

(2) Show that
$$\lim_{n\to\infty}\int_0^\infty \frac{1+nx^2+n^2x^4}{(1+x^2)^n}\,dx$$
exists, and determine its value.

Consider $\displaystyle\lim_{n\to\infty} \frac{1+nx^2+n^2x^4}{(1+x^2)^n} = \frac{\infty}{\infty}$, so use L'Hopital.

$$= \lim_{n\to\infty} \frac{x^2+2nx^4}{(1+x^2)^n \ln(1+x^2)} = \frac{\infty}{\infty} \quad \text{use L'Hopital again}$$

$$= \lim_{n\to\infty} \frac{x^4}{(1+x^2)^n(\ln(1+x^2))^2} = 0.$$

So if we can bring the limit inside, the value is 0.

Q: is the integrand dominated by some $g \in L^1((0,\infty))$?
If so, DCT tells us we can bring lim inside.

08 F. 3

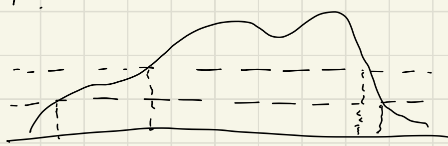
$f: \mathbb{R} \to \mathbb{R}$ bounded, by P, say, measurable function, m Lebesgue measure.
Suppose $\exists$ $M > 0$ and $c \in (0,1)$ s.t.

$$m(\{x : |f(x)| \geq t\}) \leq \frac{M}{t^c} \quad \forall\, t > 0.$$

Claim: f Lebesgue integrable. i.e. $\int_{\mathbb{R}} |f| < \infty$.

Define $f_n = |f| \chi_{\{x : |f(x)| \geq \frac{1}{n}\}}$

$$\int_{\mathbb{R}} f_n(x)\,dx = \int_{\{x : |f(x)| \geq 1/n\}} |f| \leq M n^c P.$$



$$a = m(\{x : |f(x)| \geq 1/3\}) \leq M 3^c$$
$$b = m(\{x : |f(x)| \geq 1/2\}) \leq M 2^c$$

$$m(\{x : 1/3 \leq |f(x)| \lneq 1/2\}) = \underbrace{a}_{\leq M3^c} - \underbrace{b}_{\leq M2^c} \leq M3^c - b \quad \text{also} \geq a - M2^c$$

e.g. take $M = P = 1$. $c = 1/2$ Define $f = \begin{cases} 1 & x \in [0,1) \\ 1/2 & [1, \sqrt{2}) \\ 1/3 & [\sqrt{2}, \sqrt{3}) \\ \vdots & \end{cases}$ 0, else

$$\int f = 1 + \frac{\sqrt{2}-1}{2} + \frac{\sqrt{3}-\sqrt{2}}{3} + \frac{\sqrt{4}-\sqrt{3}}{4} \cdots$$

$$\int_{\{x : |f(x)| \geq 1/n\}} |f| = \int_{\{x : 1/n \leq |f(x)| \lneq 1/n-1\}} |f| + \int_{\{x : |f(x)| \geq 1/n-1\}} |f|$$
$$\leq \frac{1}{n-1}\left(Mn^c - M(n-1)^c\right) + M(n-1)^c P$$
$$= \frac{Mn^c}{n-1} - \frac{M}{(n-1)^{1-c}} + M(n-1)^c P$$

$$\int_{|f|\geq 1} |f| \leq MP$$ max out on ≥ 1

$$\Rightarrow \int_{|f|\geq 1/2} |f| \leq MP + (M2^c - M)1 = M(P + 2^c - 1)$$

max out on each layer.

$$\Rightarrow \int_{|f|\geq 1/3} |f| \leq M(P+2^c-1) + (M3^c - M2^c)\tfrac{1}{2}$$
$$= M(P + 2^c - 1 + 3^c/2 - 2^c/2)$$

$$\Rightarrow \int_{|f|\geq 1/4} |f| \leq M(P + 2^c - 1 + 3^c/2 - 2^c/2) + (M4^c - M3^c)\tfrac{1}{3}$$
$$= M(P + 2^c - 1 + 3^c/2 - 2^c/2 + 4^c/3 - 3^c/3)$$

Does converge?

$$\int_{\mathbb{R}} |f(x)|\,dx = \int_{\mathbb{R}} \underbrace{\int_0^{|f(x)|} dt}_{= t\big]_0^{|f(x)|} = |f(x)|}\, dx \quad ? \quad \checkmark$$

08Fa.4

Define $\quad T_0^1(g) = \sup\limits_{0=x_0\leq x_1\leq\cdots\leq x_n=1} \sum_{i=1}^{n} |g(x_i)-g(x_{i-1})|$

for $g:[0,1]\to\mathbb{R}$. $\quad f_n \to f \quad \forall x\in[0,1]$, real-valued.

(i) Claim: $\quad T_0^1(f) \leq \liminf_n T_0^1(f_n)$.

That is, $\quad \sup_{\{x_i\}} \sum_{i=1}^{n} |f(x_i)-f(x_{i-1})| \leq \lim\limits_{K} \inf\limits_{n\geq K} \sup_{\{x_i\}} \sum_{i=1}^{n} |f_n(x_i)-f_n(x_{i-1})|$