Fall '08:

① μ, ν finite Borel on $\mathbb{R}^2$ such that $\mu(T) = \nu(T)$ for every open triangular region $T \subseteq \mathbb{R}^2$?
Show $\mu(E) = \nu(E)$ for all Borel sets $E \in \mathcal{B}_{\mathbb{R}^2}$

- Recall that $\mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ (Corollary 1.6), and by Prop 1.4, since $\mathcal{B}_{\mathbb{R}}$ is generated by open intervals $\{(a,b)\} = \mathcal{E}$, we have that $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ is generated by $\mathcal{F} = \{(a,b) \times (c,d) : (a,b), (c,d) \in \mathcal{E}\}$
Hence $\mathcal{B}_{\mathbb{R}^2}$ is generated by open rectangles $(a,b) \times (c,d)$.

- Now, note that any open rectangle R can be subdivided into two disjoint open triangles $T_1 \cup T_2 = R$.
Therefore:
$$\mu(R) = \mu(T_1 \cup T_2) = \mu(T_1) + \mu(T_2) = \nu(T_1) + \nu(T_2)$$
$= \nu(T_1 \cup T_2) = \nu(R)$, hence μ and ν agree on a generating set for $\mathcal{B}_{\mathbb{R}^2}$, hence they must agree on all of $\mathcal{B}_{\mathbb{R}^2}$.

(2.) Show $\lim_{n\to\infty}\int_0^\infty \frac{1+nx^2+n^2x^4}{(1+x^2)^n}\,dx$ exists, and compute

Recall that $(1+x^2)^n = \sum_{k=0}^{n}\binom{n}{k}x^{2k}$

Then we have that:

$$\frac{1+nx^2+n^2x^4}{(1+x^2)^n} = \frac{1+nx^2+n^2x^4}{1+nx+\binom{n}{2}x^4+\binom{n}{3}x^6+\cdots} \le \frac{1+nx^2+n^2x^4}{\binom{n}{3}x^6}$$

$$= \frac{6(1+nx^2+n^2x^4)}{n(n-1)(n-2)x^6} = \frac{6n^2\left(\frac{1}{n^2}+\frac{1}{n}x^2+x^4\right)}{n(n-1)(n-2)x^6}$$

$$\le \frac{6n(1+x^2+x^4)}{(n-1)(n-2)x^6}$$

Now, $\frac{d}{dn}\left[\frac{n}{(n-1)(n-2)}\right] = \frac{2-n^2}{(n-1)^2(n-2)^2} < 0$ for all $n \ge 3$,

hence $\frac{n}{(n-1)(n-2)} \le \frac{3}{(3-1)(3-2)} = \frac{3}{2}$

so $\frac{6n(1+x^2+x^4)}{(n-1)(n-2)x^6} \le \frac{9(1+x^2+x^4)}{x^6}$ for $n \ge 3$

and see that $\frac{9(1+x^2+x^4)}{x^6} \in L^1(m)$, hence we may apply DCT:

$$\lim_{n\to\infty}\int_0^\infty \frac{1+nx^2+n^2x^4}{(1+x^2)^n}\,dx = \int_0^\infty \lim_{n\to\infty}\frac{1+nx^2+n^2x^4}{(1+x^2)^n}\,dx$$

Now, note that $1+x^2>1$ for $x\in(0,\infty)$, hence $\lim_{n\to\infty}(1+x^2)^n=\infty$,

and then:

$$\lim_{n\to\infty}\frac{1}{(1+x^2)^n} = 0,$$

$$\lim_{n\to\infty}\frac{nx^2}{(1+x^2)^n} = \lim_{n\to\infty}\frac{x^2}{\ln(1+x^2)e^{n\ln(1+x^2)}} = 0$$

$$\lim_{n\to\infty}\frac{n^2x^4}{(1+x^2)^n} = \lim_{n\to\infty}\frac{2nx^4}{e^{n\ln(1+x^2)}\ln(1+x^2)} = \lim_{n\to\infty}\frac{2x^4}{e^{n\ln(1+x^2)}\ln(1+x^2)^2} = 0$$

by L'Hôpital's rule

Hence $\int_0^\infty \lim_{n\to\infty}\frac{1+nx^2+n^2x^4}{(1+x^2)^n}\,dx = \int_0^\infty 0 = 0$

(3.) $f: \mathbb{R} \to \mathbb{R}$ bounded, measurable, and m Lebesgue, and $\exists\, M > 0$, $c \in (0,1)$ s.t. $m\{|f(x)| \geq t\} \leq \frac{M}{t^c}$ for all $t > 0$.

Show f is Lebesgue integrable:

f bdd, hence $|f(x)| \leq N$ for some N.

Now, $m\{|f| \geq t\} = \int_{\mathbb{R}} \chi_{\{|f| \geq t\}}\, dm$, and we can see that:

$$\int_0^N m\{|f| \geq t\}\, dt \leq \int_0^N \frac{M}{t^c}\, dt < \infty \text{, so now:}$$

$$\int_0^N m\{|f| \geq t\}\, dt = \int_0^N \left(\int_{\mathbb{R}} \chi_{\{|f| \geq t\}}(y,t)\, dy \right) dt$$

Tonelli

$$= \int_{\mathbb{R}} \int_0^N \chi_{\{|f| \geq t\}}(y,t)\, dt\, dy$$

since $\chi_{\{|f| \geq t\}} = 1$ for $0 \leq t \leq |f(y)|$

$$= \int_{\mathbb{R}} \int_0^{|f(y)|} dt\, dy$$

$$= \int_{\mathbb{R}} |f(y)|\, dy$$

Hence $\int_{\mathbb{R}} |f|\, dm < \infty$, hence Lebesgue integrable

④ $T_0^1(g) = \sup\{\sum_{i=1}^{n}|g(x_i)-g(x_{i-1})| : 0 = x_0 < x_1 < \cdots < x_n = 1\}$ partitions

Suppose f_n, f real valued s.t. $f_n(x) \longrightarrow f(x)$ for all $x \in [0,1]$

(i) Show $T_0^1(f) \leq \liminf_{n\to\infty} T_0^1(f_n)$

(ii) If f_n abs. cont, and $T_0^1(f_n) \leq 1$ $\forall n$, then is $T_0^1(f) = \lim_{n\to\infty} T_0^1(f_n)$ → NO!

(i) $\boxed{\text{Proof \#1}}$ $f_n(x) \to f(x)$, i.e. for $x \in \mathbb{R}$, $\exists N_x$ s.t. $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$

Now: $T_0^1(f) = \sup_{P=\{x_i\}}\{\sum|f(x_i) - f(x_{i-1})|\}$

$= \sup_P\{\sum|f(x_i) - f(x_{i-1}) + f_n(x_i) - f_n(x_i) + f_n(x_{i-1}) - f_n(x_{i-1})|\}$

$\leq \sup_P\{\sum|f(x_i) - f_n(x_i)| + |f(x_{i-1}) - f_n(x_{i-1})| + |f_n(x_i) - f_n(x_{i-1})|\}$

$\leq \sup_P\{\sum|f(x_i) - f_n(x_i)|\} + \sup_P\{\sum|f(x_{i-1}) - f_n(x_{i-1})|\} + \sup_P\{\sum|f_n(x_i) - f_n(x_{i-1})|\}$

[For x_i, let $n \geq N_{x_i}^{\varepsilon} \Rightarrow |f(x_i) - f_n(x_i)| \leq \frac{\varepsilon}{2^i}$

Then let $n \geq \max_i\{N_{x_i}\}$ and we have!]

$\rightarrow \leq \sup_{k\geq 0}\{\sum_{i=0}^{k}\frac{\varepsilon}{2^i}\} + \sup_{k\geq 0}\{\sum_{i=0}^{k}\frac{\varepsilon}{2^i}\} + T_0^1(f_n)$

$\leq \varepsilon + \varepsilon + T_0^1(f_n) = 2\varepsilon + T_0^1(f_n)$

Hence $T_0^1(f) \leq 2\varepsilon + \liminf T_0^1(f_n)$, hence, letting $\varepsilon \to 0$, we get $T_0^1(f) \leq \liminf T_0^1(f_n)$!

$\boxed{\text{Proof \#2}}$ Fix part $P = \{x_0, \ldots, x_n\}$. Then

$\sum_{i=1}^{m}|f(x_i) - f(x_{i-1})| = \lim_{n\to\infty}\sum_{i=1}^{m}|f_n(x_i) - f_n(x_{i-1})| = \liminf_{n\to\infty}\sum_{i=1}^{m}|f_n(x_i) - f_n(x_{i-1})|$

$\leq \liminf_{n\to\infty} T_0^1(f_n)$.

⇒ hence, taking sup over P, get

$T_0^1(f) \leq \liminf_{n\to\infty} T_0^1(f_n)$