2008, Fall

Incomplete: 4(b).

Problem 1.

It's enough to show that μ and ν agree on open rectangles, since these generate $\mathcal{B}_{\mathbb{R}^2}$. So, suppose $R = (x_1, x_2) \times (y_1, y_2)$ is such a rectangle, and define the vectors $\boldsymbol{a} := (x_1, y_1), \boldsymbol{b} := (x_2, y_2)$. Let L be the segment $\{t\boldsymbol{a} + (1-t)\boldsymbol{b} \mid t \in (0,1)\}$, let R_1 be the open triangle with endpoints $\boldsymbol{a}, (x_1, y_2), \boldsymbol{b}$, and let R_2 be the open triangle with endpoints $\boldsymbol{a}, (x_2, y_1), \boldsymbol{b}$. Then $R = L \sqcup R_1 \sqcup R_2$, and

$$\mu(R) = \mu(L) + \mu(R_1) + \mu(R_2), \quad \nu(R) = \nu(L) + \nu(R_1) + \nu(R_2).$$

But μ and ν agree on the open triangles R_1, R_2, so we're done if we can show that $\mu(L) = \nu(L)$. Let $\boldsymbol{u}$ be a unit vector orthogonal to $\boldsymbol{b} - \boldsymbol{a}$, and for any $\epsilon > 0$, let L_ϵ be the open triangle with endpoints $\boldsymbol{a} - \epsilon\boldsymbol{u}, \boldsymbol{a} + \epsilon\boldsymbol{u}, \boldsymbol{b}$. Hence we obtain a family of open triangles $\{L_{1/j}\}_{j=1}^{\infty}$ with $\bigcap_{j=1}^{\infty} L_{1/j} = L$. Moreover, $\mu(L_1) \le \mu(\mathbb{R}^2) < \infty$ and $\nu(L_1) \le \nu(\mathbb{R}^2) < \infty$, so by continuity from above of the measures μ and ν,

$$\mu(L) = \mu\Big(\bigcap_{j=1}^{\infty} L_{1/j}\Big) = \lim_{j\to\infty} \mu(L_{1/j}) = \lim_{j\to\infty} \nu(L_{1/j}) = \nu\Big(\bigcap_{j=1}^{\infty} L_{1/j}\Big) = \nu(L),$$

since μ and ν agree on each of the open triangles $\{L_{1/j}\}_{j=1}^{\infty}$. □

Problem 2.

For fixed $x > 0$, we have

$$\lim_{n\to\infty} \frac{1 + nx^2 + n^2x^4}{(1+x^2)^n} = \lim_{n\to\infty} \frac{1}{(1+x^2)^n} + \lim_{n\to\infty} \frac{nx^2}{(1+x^2)^n} + \lim_{n\to\infty} \frac{n^2x^4}{(1+x^2)^n}.$$

The first limit is clearly 0. The second and third limits are evaluated via L'Hôspital,

$$\lim_{n\to\infty} \frac{nx^2}{(1+x^2)^n} = \lim_{n\to\infty} \frac{x^2}{\exp(n\log(1+x^2))\log(1+x^2)} = 0,$$
$$\lim_{n\to\infty} \frac{n^2x^4}{(1+x^2)^n} = \lim_{n\to\infty} \frac{2nx^4}{\exp(n\log(1+x^2))\log(1+x^2)} = 0.$$

Then provided that we can justify exchanging the limit and the integral, we have

$$\lim_{n\to\infty} \int_0^\infty \frac{1 + nx^2 + n^2x^4}{(1+x^2)^n} \mathrm{d}x = \int_0^\infty \lim_{n\to\infty} \frac{1 + nx^2 + n^2x^4}{(1+x^2)^n} \mathrm{d}x = 0.$$

To see that this is indeed justified, note that for any $x > 0$, we have

$$\frac{1 + nx^2 + n^2x^4}{(1+x^2)^n} = \frac{1 + nx^2 + n^2x^4}{\sum_{j=0}^{n} \binom{n}{j} x^{2j}} \le \frac{1 + nx^2 + n^2x^4}{\binom{n}{3} x^{2\cdot 3}} \le \frac{n}{(n-1)(n-2)} \cdot \frac{6(1 + x^2 + x^4)}{x^6}$$

by expanding and rearranging as necessary. Now when $n \ge 3$, we have

$$\frac{\mathrm{d}}{\mathrm{d}n} \frac{n}{(n-1)(n-2)} = \frac{2 - n^2}{(n-1)^2(n-2)^2} \le 0,$$

whereby the function $n/(n-1)(n-2)$ starts to decrease at $n = 3$, yielding

$$\frac{1 + nx^2 + n^2x^4}{(1 + x^2)^n} \le \frac{3}{(3-1)(3-2)} \cdot \frac{6(1 + x^2 + x^4)}{x^6} = \frac{9(1 + x^2 + x^4)}{x^6}.$$

Regarded as a function of x, the right-hand side is integrable on $(0, \infty)$, and thus we may apply dominated convergence to exchange the limit and integral above as we wished. □

Problem 3.

Let $C > 0$ be such that $|f| \le C$ a.e. Then using Fubini,

$$\|f\|_{\mathsf{L}^1(\mathbb{R})} = \int_{\mathbb{R}} |f(x)| \mathsf{d}x = \int_{\mathbb{R}} \int_0^{|f(x)|} \mathsf{d}t \mathsf{d}x = \int_{\mathbb{R}} \int_0^C \mathbb{1}_{\{|f| \ge t\}} \mathsf{d}t \mathsf{d}x = \int_0^C \int_{\mathbb{R}} \mathbb{1}_{\{|f| \ge t\}} \mathsf{d}x \mathsf{d}t$$
$$= \int_0^C m(|f| \ge t) \mathsf{d}t \le \int_0^C \frac{M}{t^c} \mathsf{d}t = \frac{MC^{1-c}}{1-c} < \infty,$$

as desired. □

Problem 4.

(a) For any $\{x_j\}_{j=0}^m \subset [0,1]$ with $0 = x_0 < x_1 < \cdots < x_m = 1$,

$$\sum_{j=1}^m |f(x_j) - f(x_{j-1})| = \liminf_{n\to\infty} \sum_{j=1}^m |f_n(x_j) - f_n(x_{j-1})| \le \liminf_{n\to\infty} T_0^1(f_n).$$

It follows that the desired inequality holds for $T_0^1(f)$, the supremum of the left-hand side over all partitions $\{x_j\}_{j=0}^m \subset [0,1]$ as above. □