Fall 2007:

① Differentiate $\longrightarrow \int_{-\infty}^{\infty} e^{-tx^2}\,dx = \sqrt{\frac{\pi}{t}}$

to show that $\int_{-\infty}^{\infty} x^{2n} e^{-tx^2}\,dx = \frac{(2n)!\sqrt{\pi}}{4^n n!}$

Consider $f(x,t) = e^{-tx^2}$ and let $t \in (\varepsilon, \infty)$ → differentiation in t and

Then $\left|\frac{\partial}{\partial t} f(x,t)\right| = \left|x^2 e^{-tx^2}\right| \le \left|\frac{x^2}{e^{\varepsilon x^2}}\right| \in L^1$

and $\left|\frac{\partial^n}{\partial t^n} f(x,t)\right| = \left|x^{2n} e^{-tx^2}\right| \le \left|\frac{x^{2n}}{e^{\varepsilon x^2}}\right| \in L^1$ for all n.

Hence apply the corollary to DCT:

$$\sqrt{\pi}\frac{d^n}{dt^n}\left(t^{-1/2}\right) = \frac{\partial^n}{\partial t^n}\int_{-\infty}^{\infty} e^{-tx^2}\,dx = \int_{-\infty}^{\infty} (-1)^n x^{2n} e^{-tx^2}\,dx$$

and $$\frac{d^n}{dt^n}\left(t^{-1/2}\right) = \left((-1)^n \frac{2n-1 \cdots 7 \cdot 5 \cdot 3}{2^n} t^{-(2n+1)/2}\right)$$

$$= \left((-1)^n \frac{(2n)!}{4^n n!} t^{-(2n+1)/2}\right)$$

Hence let $t = 1$,

and then:

$$\int_{-\infty}^{\infty} x^{2n} e^{-tx^2} = \frac{2n!}{4^n n!}\sqrt{\pi}$$

(2.) (a) Construct a sequence f_n of Leb-measurable functions on $(0,1)$ s.t. $f_n(x) \to 0$ as $n \to \infty$ for each $x \in (0,1)$ and $\int_0^1 |f_n(x)|\,dx \longrightarrow \infty$

Consider $f_n(x) = \chi_{[0,\frac{1}{n}]}(x)\,n^2$

Then $\int_0^1 |f_n(x)|\,dx = \int_0^1 \chi_{[0,\frac{1}{n}]}\,n^2 = \int_0^{1/n} n^2\,dx = n^2 \frac{1}{n} = n \longrightarrow \infty$ as $n \to \infty$

But $f_n(x) = \chi_{[0,\frac{1}{n}]}(x)\,n^2 \longrightarrow 0$ as $n \longrightarrow \infty$.

(b) Give an example of continuous fn. $F: [a,b] \to \mathbb{R}$ which is differentiable a.e. in $[a,b]$, F' Lebesgue integrable, but $F(b) - F(a) \neq \int_a^b F'(t)\,dt$

(3.) $(g_k) \subseteq L^+(E)$ where $E \subseteq \mathbb{R}^d$.

Suppose $|\{x \in E : g_k(x) > \frac{1}{2}^k\}| < \frac{1}{2^k}$ for each $k \geq 1$.

Prove that $\sum_{k=1}^{\infty} g_k$ converges almost everywhere on E

→ Let $E_k = \{x \in E : g_k(x) > \frac{1}{2^k}\}$; then $m(E_k) < \frac{1}{2^k}$ and furthermore,

$$m\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k=n}^{\infty} \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{n-1} \quad (*)$$

Hence $m\left(\bigcup_{k \geq n} E_k\right) \to 0$ as $n \to \infty$

→ Now define $B_n = \bigcup_{k \geq n} E_k$ and consider $B = \bigcap_{n=1}^{\infty} B_n$

Note that $B_1 \supseteq B_2 \supseteq \cdots$ (and $m(B_1) < \infty$), hence by cont. from above, $m(B) = m\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n \to \infty} m(B_n) = 0$ by $(*)$.

→ See now that $g_k(x) > \frac{1}{2^k}$ for infinitely many k

$\iff x \in E_k$ for infinitely many k.

$\iff x \in B_n$ for all n

$\iff x \in B$

So $x \in B$ means $g_k(x) > \frac{1}{2^k}$ for infinitely many k, but B is Lebesgue null, hence for almost every $x \in E$: $g_k(x) > \frac{1}{2^k}$ for only finitely many k, i.e. $\exists N$ such that $g_k(x) \leq \frac{1}{2^k}$ for all $k \geq N$

Hence:

$$\sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{N-1} g_k(x) + \sum_{k=N}^{\infty} g_k(x)$$

$$\leq \sum_{k=0}^{N-1} g_k(x) + \sum_{k=N}^{\infty} \frac{1}{2^k} < \infty$$ hence convergent for a.e. $x \in E$.

④ $g \in L^p(\mathbb{R}^d)$; let $\mu(t) = m(\{x : |g(x)| > t\})$ for $t \geq 0$.
Show that $\int_{\mathbb{R}^d} |g(x)|^p dx = \int_0^\infty t^p d\mu(t)$

Recall integration by parts:

$$\int_0^\infty f dg = fg\Big|_0^\infty - \int_0^\infty g df$$

Hence:

$$\int_0^\infty t^p d\mu(t) = \mu(t) t^p \Big|_0^\infty - \int_0^\infty \mu(t) d(t^p)$$

and see that

$$\int_0^\infty \mu(t) d(t^p) = \int_0^\infty \mu(t) p t^{p-1} dt = \int_0^\infty p t^{p-1} \left(\int_{\mathbb{R}^d} \chi_{\{|g(x)| > t\}} dx \right) dt$$

$$= \int_{\mathbb{R}^d} \int_0^\infty p t^{p-1} \chi_{\{|g(x)|>t\}} dt dx \quad \underline{\text{by Tonelli}}$$

$$= \int_{\mathbb{R}^d} \int_0^{|g(x)|} p t^{p-1} dt dx$$

$$= \int_{\mathbb{R}^d} |g(x)|^p dx$$

→ Thus we must only show that $\mu(t) t^p \Big|_0^\infty = 0$, i.e. that $\underline{\lim_{t\to\infty} \mu(t) t^p = 0}$.; Suppose that $g = \sum a_i \chi_{E_i}$ integrable simple.

Then for sure $M = \max\{a_i\}$, for $t \geq M$ we have:

$\mu(t) = m(\{|g(x)| \geq t\}) = 0$ [

Hence $\lim_{t\to\infty} \mu(t) t^p = 0$ (since for $t > M$, $\mu(t) t^p = 0$]

Now let ϕ_n be seq of int. simples s.t. $|\phi_1| \leq |\phi_2| \leq \dots \leq |f|$;

then $\{|\phi_i| \geq t\} \subseteq \{|\phi_{i+1}| \geq t\} \subseteq \{|f| \geq t\}$ for all i,

hence by cont from below, $\lim_{i\to\infty} m\{|\phi_i| \geq t\} = \{|f| \geq t\}, = \mu(t)$

hence $\lim_{t\to\infty} \mu(t) t^p = \lim_{t\to\infty} t^p \lim_{i\to\infty} m\{|\phi_i| \geq t\} = \lim_{i\to\infty} 0 = 0$