2007, Fall

Problem 1.

Let $n \in \mathbb{N}$ and $t > 0$. Choose $\epsilon > 0$ small enough so that $t > \epsilon$. By dominated convergence, we may move the operator $\mathsf{d}^n/\mathsf{d}t^n$ inside the given integral since

$$\left|\frac{\mathsf{d}^n}{\mathsf{d}t^n} e^{-tx^2}\right| = \left|(-1)^n x^{2n} e^{-tx^2}\right| \le \left|x^{2n} e^{-\epsilon x^2}\right|,$$

and the right-hand side, regarded as a function of x on $\mathbb{R}$, is integrable. Hence

$$\int_{-\infty}^{\infty} (-1)^n x^{2n} e^{-tx^2} \mathsf{d}x = \int_{-\infty}^{\infty} \frac{\mathsf{d}^n}{\mathsf{d}t^n} e^{-tx^2} \mathsf{d}x = \frac{\mathsf{d}^n}{\mathsf{d}t^n} \sqrt{\frac{\pi}{t}} = \sqrt{\pi} \cdot \frac{(-1)^n (2n)!}{4^n n!} t^{-(2n+1)/2},$$

whereby setting $t := 1$ gives

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2} \mathsf{d}x = \frac{(2n)!\sqrt{\pi}}{4^n n!},$$

as desired. □

Problem 2.

(a) Set $f_j := j^2 \mathbb{1}_{(0,j^{-1})}$ for each $j \in \mathbb{N}$. Then

$$\lim_{j\to\infty} \int_{(0,1)} f_j = \lim_{j\to\infty} \int_{(0,j^{-1})} j^2 = \lim_{j\to\infty} j = \infty.$$

However, for any fixed $x \in (0,1)$, for all $j \in \mathbb{N}$ sufficiently large, we have $j^{-1} < x$ and so $f_j(x) = 0$. Thus $\lim_{j\to\infty} f_j(x) = 0$. □

(b) Let $f : [0,1] \to [0,1]$ be the well-known Devil's staircase function. Then f increases continuously from $f(0) = 0$ to $f(1) = 1$. But outside of the measure-0 Cantor set, f' exists and is identically 0, so $f(1) - f(0) = 1 \neq 0 = \int_0^1 f'(x)\mathsf{d}x$. □

Problem 3.

Set $E_j := \{g_j > 2^{-j}\}$ for each $j \in \mathbb{N}$. If $x \in E_j$ for only finitely many $j \in \mathbb{N}$, then there's some $N \in \mathbb{N}$ so that $x \in E_j^{\mathsf{c}}$ for all $j \ge N$, and hence the sum converges for this x,

$$\sum_{j=1}^{\infty} g_j(x) = \sum_{j=1}^{N-1} g_j(x) + \sum_{j=N}^{\infty} g_j(x) < \underbrace{\sum_{j=1}^{N-1} g_j(x)}_{<\infty} + \underbrace{\sum_{j=N}^{\infty} \frac{1}{2^j}}_{<\infty} < \infty.$$

Hence we're done if we can show that the set of those x's belonging to infinitely many E_j's is a null set. This is precisely the set $\limsup_{j\to\infty} E_j$, and we have

$$\mu\left(\limsup_{j\to\infty} E_j\right) = \mu\Big(\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j\Big) = \lim_{k\to\infty} \mu\Big(\bigcup_{j=k}^{\infty} E_j\Big) \le \lim_{k\to\infty} \sum_{j=k}^{\infty} \mu(E_j).$$

But each summand on the right is bounded above by 2^{-j}, and the sum $\sum_{j=1}^{\infty} 2^{-j}$ is convergent, whereby the limit on the right is 0. □

Problem 4 (?)**.**

Set $E_t := \mu(\{|g| > t\})$. Integrating by parts,

$$\int_0^\infty \mu(t)\mathsf{d}(t^p) + \int_0^\infty t^p \mathsf{d}\mu(t) = \mu(t)t^p\Big|_0^\infty = \lim_{t\to\infty} \mu(t)t^p.$$

By Fubini, the first integral is equal to

$$\int_0^\infty \left(\int_{\mathbb{R}^d} \mathbb{1}_{E_t}\mathsf{d}x\right) pt^{p-1}\mathsf{d}t = \int_{\mathbb{R}^d}\int_0^\infty \mathbb{1}_{E_t} pt^{p-1}\mathsf{d}t\mathsf{d}x = \int_{\mathbb{R}^d}\int_0^{|g(x)|} pt^{p-1}\mathsf{d}t = \int_{\mathbb{R}^d} |g(x)|^p \mathsf{d}x.$$

Thus the result follows if we can show that $\lim_{t\to\infty} \mu(t)t^p = 0$. Let $\{\varphi_j\}_{j=1}^\infty \subset \mathsf{L}^p(\mathbb{R}^d)$ be a sequence of nonnegative simple functions approaching g with $|\varphi_1| \le |\varphi_2| \le \cdots \le |g|$ a.e. Then for any $t \ge 0$,

$$\{|\varphi_1| > t\} \subset \{|\varphi_2| > t\} \subset \cdots \subset \{|g| > t\} = E_t, \quad E_t = \bigcup_{j=1}^\infty \{|\varphi_j| > t\}.$$

For any $j \in \mathbb{N}$, writing $\varphi_j = \sum_{k=1}^m a_k \mathbb{1}_{A_k}$ for some $a_k \ge 0$ and $A_k \in \mathcal{M}$, the set $\{|\varphi_j| > t\}$ has measure 0 as soon as $t > \max_{1\le k\le m} a_k$, whereby

$$\lim_{t\to\infty} \mu(t)t^p = \lim_{t\to\infty}\lim_{j\to\infty} \mu(\{|\varphi_j| > t\})t^p = 0.$$

□