06 Fa.1

1. Find necessary and sufficient conditions for a subset $X \subset \mathbb{R}$ to belong to the σ-algebra generated by all one-point subsets of $\mathbb{R}$.

Let $\mathcal{M}$ denote the σ-algebra generated by all one-point sets.
Let $\mathcal{N} = \{X : X \text{ or } X^c \text{ countable}\}$, a set.

Claim: $\mathcal{M} = \mathcal{N}$.

First, we prove $\mathcal{N} \subset \mathcal{M}$. Indeed, if X countable, then $X = \bigcup_1^\infty \{x_i\} \Rightarrow X \in \mathcal{M}$. And if X^c countable, $X^c \in \mathcal{M}$ $\Rightarrow X \in \mathcal{M}$.

Second, we prove $\mathcal{M} \subset \mathcal{N}$. If $\mathcal{N}$ is a σ-algebra, then $\mathcal{M} \subset \mathcal{N}$ since $\mathcal{M}$ is the smallest σ-algebra containing all one-point sets.

$\mathcal{N}$ is closed under complement: Let $X \in \mathcal{N}$. If X countable then $X^c \in \mathcal{N}$ since $X = (X^c)^c$ is countable. Otherwise, X^c countable $\Rightarrow$ $X^c \in \mathcal{N}$.

$\mathcal{N}$ is closed under countable union: Let $\{X_i\}_1^\infty \subset \mathcal{N}$.
If X_i countable $\forall i$ then $\bigcup X_i$ countable $\Rightarrow \bigcup X_i \in \mathcal{N}$.
If not, then for some i, X_i^c countable.
$\Rightarrow (\bigcup X_i)^c = \bigcap X_i^c$ countable $\Rightarrow \bigcup X_i \in \mathcal{N}$.

$\square$

06 Fa. 2

2. Let (X, μ) be a measure space. Which of the following implications are true?

a. $\mu(X) < \infty$ and $f \in L^2(\mu)$ implies $f \in L^1(\mu)$.

b. $\mu(X) = \infty$ and $f \in L^2(\mu)$ implies $f \in L^1(\mu)$.

c. $\mu(X) < \infty$ and $f \in L^1(\mu)$ implies $f \in L^2(\mu)$.

d. $\mu(X) = \infty$ and $f \in L^1(\mu)$ implies $f \in L^2(\mu)$.

Give proof or counter-example in each case.

06 Fa. 3

3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} e^{-(x-y)} & \text{if } x > y, \\ 0 & \text{if } x = y, \\ -e^{-(y-x)} & \text{if } x < y. \end{cases}$$

a. Is f Lebesgue integrable?

b. Is it true that

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x,y)\, dx \right) dy = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x,y)\, dy \right) dx \ ?$$

06 Fa. 4

4. Let $f \in L^1(\mathbb{R})$. Show that for each $n = 1, 2, 3, \ldots$, the function

$$f_n(x) = f(x)(\sin x)^n$$

also belongs to $L^1(\mathbb{R})$ and that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = 0.$$