Fall 06

(1.) Find necessary and sufficient conditions for $X \subseteq \mathbb{R}$ to belong to σ-algebra gen. by 1-pt subsets of $\mathbb{R}$

Let $\mathcal{A} = \{E \subseteq X;\ E \text{ countable or } E^c \text{ countable}\}$.

- $\mathcal{A}$ is a σ-algebra :
  1. Countable additivity : let $(E_i) \subseteq \mathcal{A}$.
  Consider $E = \bigcup_{i=1}^{\infty} E_i$. If all the E_i are countable, then $E \in \mathcal{A}$. Suppose that some E_j is cocountable.
  Then $E^c = \left(\bigcup_{i=1}^{\infty} E_i\right)^c = \bigcap_{i=1}^{\infty} E_i^c \subseteq E_j^c$, which is countable, hence E^c is countable, hence $E \in \mathcal{A}$.
  2. Complements : Consider $E, F \in \mathcal{A}$. If both countable, then $E \setminus F \subseteq E$ countable, hence $E \setminus F \in \mathcal{A}$. Suppose E countable and F cocountable. Then $E \setminus F \subseteq E$ countable $\Rightarrow E \setminus F \in \mathcal{A}$, and $(F \setminus E)^c = (F \cap E^c)^c = F^c \cup E$, countable, hence $F \setminus E \in \mathcal{A}$.
  Suppose both cocountable; then $F \setminus E = F \cap E^c \subseteq E^c$ hence countable, hence $F \setminus E \in \mathcal{A}$.

- $P = \{\text{singleton sets}\} \subseteq \mathcal{P}(\mathbb{R})$; show $\mathcal{M}(P) = \mathcal{A}$.

$\mathcal{M}(P)$: σ-algebra generated by P

$(\subseteq)$ $P \subseteq \mathcal{A} \Rightarrow \mathcal{M}(P) \subseteq \mathcal{A}$.

$(\supseteq)$ $E \in \mathcal{A}$;

If E countable, then $E = \bigcup_{n=1}^{\infty} \{x_i\} \in \mathcal{M}(P)$

If E cocountable, then E^c countable, then $E^c = \bigcup_{n=1}^{\infty} \{y_i\} \in \mathcal{M}(P)$ and $E = (E^c)^c \in \mathcal{M}(P)$.

So $\mathcal{A} \subseteq \mathcal{M}(P)$,

Therefore, $\mathcal{M}(P) = \mathcal{A}$

Hence $X \in \mathcal{M}(P) \iff X$ countable or cocountable

(2.) (X,μ) measure space. True or false?

(a) $\mu(X)<\infty$ and $f\in L^2(\mu) \Rightarrow f\in L^1(\mu)$: TRUE

$\|f\|_1 = \|f\cdot 1\|_1 \le \|f\|_2 \|1\|_2$ by Hölder's inequality

$f\in L^2(\mu) \Rightarrow \|f\|_2<\infty$ and $\mu(X)<\infty \Rightarrow \|1\|_2<\infty$.

Hence $\|f\|_1<\infty$, hence $f\in L^1(\mu)$.

(b) $\mu(X)=\infty$ and $f\in L^2(\mu) \Rightarrow f\in L^1(\mu)$: FALSE

$X=[1,\infty)$, $\mu(X)=\infty$, $f(x)=\frac{1}{x}$.

Then $f\in L^2$ since $\int_1^\infty \frac{1}{x^2}dx<\infty$, but $f\notin L^1$ since $\int_1^\infty \frac{1}{x}dx=\infty$

(c) $\mu(X)<\infty$ and $f\in L^1(\mu) \Rightarrow f\in L^2(\mu)$: FALSE

$X=[0,1]$, $\mu(X)<\infty$, $f(x)=\frac{1}{\sqrt{x}}$.

Then $\int_0^1 |f|dx<\infty$, but $\int_0^1 |f|^2dx = \int_0^1 \frac{1}{x}dx = \infty$.

(d) $\mu(X)=\infty$ and $f\in L^1(\mu) \Rightarrow f\in L^2(\mu)$ FALSE

$X=[0,\infty)$, $\mu(X)=\infty$; $f(x)=\frac{1}{\sqrt{x}}\chi_{(0,1]}$.

Then $\int_0^\infty |f|dx = \int_0^1 \frac{1}{\sqrt{x}}dx<\infty$, but $\int_0^\infty |f|^2dx = \int_0^1 \frac{1}{x}dx = \infty$

Alt proof of (a) w/o Hölder:

$\mu(X)<\infty$ and $f\in L^2(\mu)$. Let $E=\{|f|>1\}$.

Then $\|f\|_1 = \int_X |f|d\mu = \int_E |f|d\mu + \int_{X\setminus E}|f|d\mu$

$\le \int_E |f|^2 d\mu + \int_{X\setminus E} d\mu < \infty$

since $f\in L^2(\mu)$ and

$\mu(X\setminus E)<\infty$ since $\mu(X)<\infty$

(3.) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = \begin{cases} e^{-(x-y)} & \text{if } x>y \\ 0 & \text{if } x=y \\ -e^{-(y-x)} & \text{if } x<y \end{cases}$

(a) Is f Lebesgue integrable?
(b) Is it true that: $\iint f(x,y)\,dx\,dy = \iint f(x,y)\,dy\,dx$

(a) See that $|f(x,y)| = |f(y,x)|$, so we have:

$$\iint |f|\,dm = 2\iint_{\{x>y\}} |f|\,dm = 2\int_{\infty}^{\infty}\left[\int_{-\infty}^{x} e^{y-x}\,dy\right]dx \quad (*)$$

Now, see that $\int_{-\infty}^{x} e^{y-x}\,dy = \int_{-\infty}^{0} e^{u}\,du$ (under substitution $u=y-x$)

$$= e^{0} - \lim_{u\to-\infty} e^{u} = 1 - 0 = 1$$

So:

$(*) = 2\int_{-\infty}^{\infty} 1\,dx = \infty$, hence f is not Lebesgue integrable.

(b) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\,dx\,dy = \int_{\infty}^{\infty}\left(\int_{\{x>y\}} e^{-(x-y)}dx + \int_{\{x<y\}} -e^{-(y-x)}dx\right)dy$$

$$= \int_{-\infty}^{\infty}\left(\int_{\{x>y\}} e^{-(x-y)}dx - \int_{\{x<y\}} e^{-(y-x)}dx\right)dy = \int_{\infty}^{\infty}\left(\int_{\{x>y\}} e^{-(x-y)}\cancel{dx} \overset{0}{-} \int_{\{x>y\}} e^{-(x-y)}\right)dy$$

$= \int_{-\infty}^{\infty} 0\,dy = 0$, and same for the other side,

hence the equality holds

(4) $f \in L^1(\mathbb{R})$. Show $f_n(x) = f(x)\sin^n x \in L^1(\mathbb{R})$ for all n, and that $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = 0$

See that $|f_n(x)| = |f(x)\sin^n x| = |f|\,|\sin x|^n \leq |f| \in L^1$, hence $f_n \in L^1$.

Now, $\lim_{n\to\infty} \int |f_n|\,dx = \int \lim_{n\to\infty} |f_n|\,dx$ by DCT since $|f_n| \leq |f|$ for all n and $f \in L^1$.

Let $x \in \mathbb{R} \setminus \{\frac{(2k+1)\pi}{2}\}_{k\in\mathbb{Z}}$; then for all such x, $|\sin x| < 1$, hence $\lim_{n\to\infty} |\sin x|^n = 0$ on $\mathbb{R} \setminus \{\frac{(2k+1)\pi}{2}\}$,

hence $\lim_{n\to\infty} |f(x)\sin^n x| = |f(x)| \lim_{n\to\infty} |\sin x|^n = 0$

for all such x, hence $\lim_{n\to\infty} \int |f_n|\,dx = 0$ for all such x.

Now, the set for which this not true $\left(\{\frac{(2k+1)\pi}{2}\}_{k\in\mathbb{Z}}\right)$ is countable, hence measure 0,

hence $\lim_{n\to\infty} \int |f_n|\,dx = 0$ almost everywhere.