2006, Fall

Problem 1.

Let S be the collection of all 1-point subsets of $\mathbb{R}$, and $\sigma(S)$ the σ-algebra generated by S. Now let $\mathcal{F} := \{E \subset \mathbb{R} \mid E \text{ is countable or cocountable}\}$ (it's easy to show that $\mathcal{F}$ is a σ-algebra). We claim that $E \in \sigma(S)$ if and only if $E \in \mathcal{F}$. The inclusion $S \subset \mathcal{F}$ is immediate, so $\sigma(S) \subset \mathcal{F}$. Conversely if $E \in \mathcal{F}$ is countable (resp. cocountable), then it's a countable union (resp. complement of a countable union) of 1-point subsets, and hence $E \in \sigma(S)$; so $\mathcal{F} \subset \sigma(S)$. □

Problem 2.

(a) **True**. By Hölder, $\|f\|_{\mathsf{L}^1(\mu)} \leq \|f\|_{\mathsf{L}^2(\mu)}\|1\|_{\mathsf{L}^2(\mu)} = \|f\|_{\mathsf{L}^2(\mu)}\mu(X)^{1/2} < \infty$. □

(b) **False**. Set $X := (1, \infty)$ with Lebesgue measure μ, and $f(x) := x^{-1}$. Then

$$\|f\|_{\mathsf{L}^1(\mu)} = \int_1^\infty x^{-1}\mathsf{d}x = \infty, \quad \|f\|_{\mathsf{L}^2(\mu)} = \left(\int_1^\infty x^{-2}\mathsf{d}x\right)^{1/2} = 1 < \infty.$$

□

(c) **False**. Set $X := (0, 1)$ with Lebesgue measure μ, and $f(x) := x^{-1/2}$. Then

$$\|f\|_{\mathsf{L}^1(\mu)} = \int_0^1 x^{-1/2}\mathsf{d}x = 2 < \infty, \quad \|f\|_{\mathsf{L}^2(\mu)} = \left(\int_0^1 x^{-1}\mathsf{d}x\right)^{1/2} = \infty.$$

□

(d) **False**. Extend the function f in (c) to all of $X := \mathbb{R}$ by setting $f :\equiv 0$ outside of $(0, 1)$. □

Problem 3 (?)**.**

(a) **No**. We have $|f(x, y)| = |f(y, x)|$ for any $(x, y) \in \mathbb{R}^2$, and so by symmetry

$$\|f\|_{\mathsf{L}^1(\mathbb{R}^2)} = \iint_{\mathbb{R}^2} |f| = 2\iint_{\{x>y\}} |f(x, y)|\mathsf{d}y\mathsf{d}x = 2\int_{-\infty}^\infty \underbrace{\int_{-\infty}^x e^{-(x-y)}\mathsf{d}y}_{=1}\, \mathsf{d}x = \infty$$

(the inner integral is equal to 1 by an easy computation). □

(b) **Yes**. Both integrals are equal to 0 by substitution. □

Problem 4.

The function $|f|$ is in $\mathsf{L}^1(\mathbb{R})$, and for each $n \in \mathbb{N}$ we have $|f_n| = |f| \cdot |\mathsf{sin}(x)|^n \leq |f|$, hence

$$\|f_n\|_{\mathsf{L}^1(\mathbb{R})} = \int_{\mathbb{R}} |f_n| \leq \int_{\mathbb{R}} |f| = \|f\|_{\mathsf{L}^1(\mathbb{R})} < \infty.$$

Now $|\mathsf{sin}(x)| < 1$ for a.e. $x \in \mathbb{R}$, so $\mathsf{lim}_{n\to\infty} f_n = 0$ a.e. Then

$$\lim_{n\to\infty} \int_{-\infty}^\infty f_n(x)\mathsf{d}x = \int_{-\infty}^\infty 0\, \mathsf{d}x = 0$$

by dominated convergence. □