Fall 2005:

(1.) $(X, \mathcal{B}, \mu)$ meas. space w/ $\mu(X) < \infty$. For $A \subseteq X$, define:

$$\mu_1(A) = \sup_{B \in \mathcal{B}} \{\mu(B) : B \subseteq A\}, \quad \mu_2(A) = \inf_{B \in \mathcal{B}} \{\mu(B) : B \supseteq A\}$$

(a) Show $\mu_1(A^c) + \mu_2(A) = \mu(X)$

(b) $\mathcal{A} = \{A \subseteq X : \mu_1(A) = \mu_2(A)\}$ a σ-algebra

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(a):
$$\begin{aligned}
\mu_1(A^c) &= \sup_{B \in \mathcal{B}} \{\mu(B) : B \subseteq A^c\} \\
&= \sup_{B \in \mathcal{B}} \{\mu(B) : B^c \supseteq A\} \\
&= \sup_{B \in \mathcal{B}} \{\mu(X) - \mu(B^c) : B^c \supseteq A\} \\
&= \mu(X) - \inf_{B \in \mathcal{B}} \{\mu(B^c) : B^c \supseteq A\} \\
&= \mu(X) - \inf_{C \in \mathcal{B}} \{\mu(C) : C \supseteq A\} \\
&= \mu(X) - \mu_2(A)
\end{aligned}$$

since $\mathcal{B}$ closed under complements

$\Rightarrow \underline{\mu(X) = \mu_1(A^c) + \mu_2(A)}$.

(b) ~~Let~~ $(A_i) \subseteq \mathcal{A}$, so $\mu_1(A_i) = \mu_2(A_i)$; First see that

$$\mu_1(A) = \sup_{B \subseteq A} \{\mu(B)\} \leq \mu(A) \leq \inf_{A \subseteq B} \{\mu(B)\} = \mu_2(A)$$

Hence need only show $\mu_1(\cup A_i) \geq \mu_2(\cup A_i)$

$$\begin{aligned}
\mu_2(\cup A_i) &= \inf_{B \in \mathcal{B}} \{\mu(B) : B \supseteq \cup A_i\} \\
&\leq \inf_{B \in \mathcal{B}} \{\mu(B) : B \supseteq \cup A_i,\ B = \cup B_i,\ A_i \subseteq B_i\} \\
&\leq \sum_i \inf_{B} \{\mu(B_i) : A_i \subseteq B_i\} = \sum_i \mu_2(A_i) = \sum_i \mu_1(A_i) \\
&= \sum_i \sup_{C \in \mathcal{B}} \{\mu(C_i) : C_i \subseteq A_i\} \\
&\leq \sup_{C \in \mathcal{B}} \{\mu(C) : C \subseteq \cup A_i\} = \mu_1(\cup A_i)
\end{aligned}$$

(2) μ meas. on $[0,\infty)$ s.t. $\int_{[0,\infty)} e^{ax}\, d\mu < \infty$ for $a \in \mathbb{R}$.
Show that $\psi(t) = \int_{[0,\infty)} e^{tx}\, d\mu$ is infinitely diff. on $(-\infty, a)$.

Let $f(x,t) = e^{tx}$ and consider $t \in (-\infty, a-\varepsilon)$.

Then $\left|\frac{\partial f}{\partial t}(x,t)\right| = \left|x e^{tx}\right| \leq x e^{(a-\varepsilon)x} = \frac{x e^{ax}}{e^{\varepsilon x}} \leq M e^{ax}$

since $\frac{x}{e^{\varepsilon x}}$ is a bounded function on $[0,\infty)$ (with bd M)

Clearly $M e^{ax}$ is in $L^1(\mu)$, hence we may apply the corollary to DCT.

(3.) $(X, \mathcal{B}, \mu)$ meas space, $0 \le f < \infty$ meas, $d\nu = f\,d\mu$

(a) Suppose μ σ-finite. Show ν σ-finite:

Since μ σ-finite, we may take the Lebesgue decomposition of ν w.r.t. μ: $\exists \lambda$ meas and $g \in L^1(\mu)$ s.t. $\lambda \perp \mu$ and $d\nu = d\lambda + g\,d\mu$

Then $f\,d\mu = d\lambda - g\,d\mu \Rightarrow f\,d\mu - g\,d\mu = d\lambda$
$\Rightarrow (f-g)\,d\mu = d\lambda.$

Since $\lambda \perp \mu$, $\exists$ E s.t. $\lambda(E) = 0$ and $\mu(E^c) = 0$, so:

$$\lambda(A) = \lambda((A \cap E) \cup (A \cap E^c))$$
$$= \lambda(A \cap E) \cup \lambda(A \cap E^c) = 0 + \int_{A \cap E^c} (f-g)\,d\mu = 0 + 0$$
since $\mu(E^c) = 0$.

So λ identically 0; hence $f\,d\mu = g\,d\mu$,

Now since μ σ-finite, $\exists$ $X = \bigcup_i X_i$ s.t. $\mu(X_i) < \infty$.

But now: $\nu(X_i) = \int_{X_i} g\,d\mu < \infty$ since $g \in L^1(\mu)$

(b) Show that if $0 < f < \infty$ and ν semi-finite, then μ semifinite

Recall that if ν semifinite then for $\nu(E) > 0$, $\exists F \subseteq E$ s.t. $0 < \nu(F) < \infty$; namely:

$$\nu(E) = \int_E f\,d\mu > 0 \Rightarrow \exists F \subseteq E \text{ s.t. } \nu(F) = \int_F f\,d\mu < \infty$$

But since $f > 0$, $\nu(E) = \int_E f\,d\mu > 0$ for all E with $\mu(E) > 0$;

so if $\mu(E) > 0$, $\exists F \subseteq E$ w/ $0 < \nu(F) = \int_F f\,d\mu < \infty$

$\Rightarrow 0 < \mu(F) < \infty$ since $f > 0$ on all of F
(otherwise, integral will diverge)

hence μ semifinite

④ $(X, \mathcal{B}, \mu)$ with $\mu(X) < \infty$, f_n integrable, $f_n \to f$ in L^1

(a) Show $f_n \to f$ in measure

(b) Show $d\nu_n = |f_n| d\mu$ are unif. abs. cont. w.r.t μ.

(a) See that $\int_{\{|f_n - f| \geq \varepsilon\}} |f_n - f| d\mu \geq \int_{\{|f_n - f| \geq \varepsilon\}} \varepsilon \, d\mu = \varepsilon \mu\{|f_n - f| \geq \varepsilon\}$.

$\Rightarrow \mu\{|f_n - f| \geq \varepsilon\} \leq \frac{1}{\varepsilon} \int_{\{|f_n - f| \geq \varepsilon\}} |f_n - f| d\mu < \frac{1}{\varepsilon} \int_X |f_n - f| d\mu \quad (*)$

Now, for $\varepsilon\varepsilon' > 0$, $\exists M$ s.t. $n \geq M \Rightarrow \int_X |f_n - f| < \varepsilon\varepsilon'$,

hence for large n, $(*) < \frac{1}{\varepsilon} \varepsilon\varepsilon' = \varepsilon'$,

hence for large n, $\mu\{|f_n - f| \geq \varepsilon\} < \varepsilon'$, hence conv. in measure.

(b) $\left[\begin{array}{l}\text{Recall unif. abs. cont. w.r.t. } \mu \text{ means:} \\ \text{for } \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \mu(E) < \delta \Rightarrow \nu_n(E) < \varepsilon \text{ for all } n\end{array}\right]$

By definition, $\nu_n \ll \mu$, hence for $\frac{\varepsilon}{2^n}$, $\exists \delta > 0$ such that $\mu(E) < \delta \Rightarrow \nu_n(E) < \frac{\varepsilon}{2^n}$.

Therefore, for all n (since ν_n is positive)

$$\nu_n(E) \leq \sum_n \nu_n(E) < \sum_n \frac{\varepsilon}{2^n} < \varepsilon$$