REAL ANALYSIS GRADUATE EXAM
Fall, 2005

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $(X, \mathcal{B}, \mu)$ be a measure space with μ finite. For all $A \subset X$ define

$$\mu_1(A) = \sup\{\mu(B) : B \in \mathcal{B}, B \subset A\}, \qquad \mu_2(A) = \inf\{\mu(B) : B \in \mathcal{B}, B \supset A\}.$$

(a) Show that $\mu_1(A^c) + \mu_2(A) = \mu(X)$ for all $A \subset X$.

(b) Let $\mathcal{A} = \{A \subset X : \mu_1(A) = \mu_2(A)\}$. Show *directly from (a) and/or the definitions* that $\mathcal{A}$ is a σ-algebra.

(2) Suppose μ is a measure on $[0, \infty)$ satisfying

$$\int_{[0,\infty)} e^{ax}\ \mu(dx) < \infty \quad \text{for some } a \in \mathbb{R}.$$

Show that the function

$$\psi(t) = \int_{[0,\infty)} e^{tx}\ \mu(dx)$$

is infinitely differentiable on $(-\infty, a)$.

(3) Suppose $(X, \mathcal{B}, \mu)$ is a measure space, $0 \le f < \infty$ is a measurable function, and $d\nu = f d\mu$.

(a) Suppose μ is σ-finite. Show that ν is σ-finite. HINT: You must deal with the fact that f is not assumed bounded or integrable.

(b) A measure ρ is called *semifinite* if for every measurable set E with $\rho(E) > 0$, there is a measurable $F \subset E$ with $0 < \rho(F) < \infty$. Show that if $0 < f < \infty$ and ν is semifinite, the μ is semifinite. (We do not keep the assumption made in (a) that μ is σ-finite.)

(4) Suppose $(X, \mathcal{B}, \mu)$ is a measure space with $\mu(X)$ finite, $\{f_n\}$ are integrable functions, and $f_n \to f$ in L^1.

(a) Show that $f_n \to f$ in measure.

(b) Show that the measures $d\nu_n = |f_n|\ d\mu$ are uniformly absolutely continuous with respect to μ. NOTE: This means that for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\nu_n(E) < \epsilon$ for all n. Absolute continuity of each ν_n *individually* is a standard result which you may make use of.