Fall 04:

(1.) $f:[a,b]\to\mathbb{R}$ continuous. Show: $\lim_{n\to\infty}\|f\|_{L^n} = \sup_{x\in[a,b]}|f(x)| = \|f\|_\infty$

For $\varepsilon>0$, let $S_\varepsilon := \{|f| \geq \|f\|_\infty - \varepsilon\} \subseteq [a,b]$

Hence, $\|f\|_{L^n} = \left(\int_a^b |f(x)|^n dx\right)^{1/n} \geq \left(\int_{S_\varepsilon} |f(x)|^n dx\right)^{1/n} \geq \left(\int_{S_\varepsilon} (\|f\|_\infty - \varepsilon)^n dx\right)^{1/n}$

$= \left((\|f\|_\infty-\varepsilon)^n \mu(S_\varepsilon)\right)^{1/n} = (\|f\|_\infty - \varepsilon)\mu(S_\varepsilon)^{1/n}$

$\Rightarrow \liminf_{n\to\infty}\|f\|_n \geq \liminf_{n\to\infty}(\|f\|_\infty-\varepsilon)\mu(S_\varepsilon)^{1/n} = (\|f\|_\infty - \varepsilon)\lim_{n\to\infty}\mu(S_\varepsilon)^{1/n}$ $\boxed{= 1 \text{ since } \mu(S_\varepsilon) > 0}$

$= (\|f\|_\infty - \varepsilon)$

$\Rightarrow \liminf_{n\to\infty}\|f\|_n \geq \|f\|_\infty - \varepsilon$

Hence letting $\varepsilon\to 0$ we get $\underline{\liminf_{n\to\infty}\|f\|_n \geq \|f\|_\infty}$ (1)

On the other hand, $|f(x)| \leq \|f\|_\infty$ for all x, and for $p>q$,

$$\|f\|_{L^p} = \left(\int |f(x)|^p dx\right)^{1/p} = \left(\int |f(x)|^{p-q}|f(x)|^q d\mu\right)^{1/p}$$
$$\leq \left(\|f\|_\infty^{p-q}\int |f(x)|^q d\mu\right)^{1/p}$$
$$= \|f\|_\infty^{\frac{p-q}{p}}\|f\|_{L^q}^{q/p}$$

$(n>q)$

$\Rightarrow \underline{\limsup_{n\to\infty}\|f\|_{L^n}} \leq \limsup_{n\to\infty}\left(\|f\|_\infty^{\frac{n-p}{n}}\|f\|_{L^p}^{\frac{p}{n}}\right) = \limsup_{n\to\infty}\|f\|_\infty\left(\frac{\|f\|_{L^p}}{\|f\|_\infty}\right)^{p/n}$ (2)

$= \|f\|_\infty$ since $\|f\|_{L^p}/\|f\|_\infty > 0$

Hence: $\|f\|_\infty \overset{(1)}{\leq} \liminf_{n\to\infty}\|f\|_{L^n} \leq \lim_{n\to\infty}\|f\|_{L^n} \leq \limsup_{n\to\infty}\|f\|_{L^n} \underset{(2)}{\leq} \|f\|_\infty$

$\Rightarrow \underline{\lim_{n\to\infty}\|f\|_{L^n} = \|f\|_\infty}$

(2) $f: \mathbb{R} \to \mathbb{R}$ integrable. Prove that $g(x) = \sum_{n=1}^{\infty} f(2^n x + \frac{1}{n})$ is integrable and $\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$

Let $g_k(x) = \sum_{n=1}^{k} |f(2^n x + \frac{1}{n})|$.

Now:

$$\int_{\mathbb{R}} |g(x)|\,dx = \int \left|\sum_{n=1}^{\infty} f(2^n x + \tfrac{1}{n})\right| dx = \int \lim_{k\to\infty} \left|\sum_{n=1}^{k} f(2^n x + \tfrac{1}{n})\right| dx$$

$$\leq \int \lim_{k\to\infty} \sum_{n=1}^{k} |f(2^n x + \tfrac{1}{n})| = \int \lim_{k\to\infty} g_k(x) \quad (*)$$

Now, note that $g_k(x) \leq g_{k+1}(x)$ since $|f(2^n x + \frac{1}{n})| \geq 0$, for all n.

Hence since the g_k are measurable, we may apply MCT.

$$(*) = \lim_{k\to\infty} \int g_k(x) = \lim_{k\to\infty} \int \sum_{n=1}^{k} |f(2^n x + \tfrac{1}{n})|\,dx = \lim_{k\to\infty} \sum_{n=1}^{k} \int |f(2^n x + \tfrac{1}{n})|\,dx$$

$$= \sum_{n=1}^{\infty} \int \frac{|f(u)|\,du}{2^n} = \int |f(u)|\,du \left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right) = \int |f(u)|\,du$$

$$\begin{cases} u = 2^n x + \frac{1}{n} \\ du = 2^n dx \end{cases}$$

③ f Leb. integrable on $[0,4]$; $\int_E f = 0$ $\forall E$ with $m(E) = \pi$

Show $f = 0$ a.e. :

Consider the sets $E^+ = \{f > 0\}$, $E^- = \{f < 0\}$, $E^0 = \{f = 0\}$

these are all measurable since f is measurable.

Now, $\mu(E^+ \cup E^0) < \pi$ and $\mu(E^- \cup E^0) < \pi$, otherwise they will have a subset F that has $\int_F f = 0$, which is impossible.

Since $\mu(E^+ \cup E^0) < \pi$, $\exists F^- \subseteq E^-$ s.t. $\mu(E^+ \cup E^0 \cup F^-) = \pi$

$\Rightarrow \int_{E^+ \cup E^0 \cup F^-} f = 0 \Rightarrow \boxed{\int_{E^+ \cup E^0} f = a}$ and $\boxed{\int_{F^-} f = -a}$

Now, on the other hand, since $\mu(E^- \cup E^0) < \pi$, $\exists F^+ \subseteq E^+$ s.t. $\mu(F^+ \cup E^0 \cup E^-) = \pi$.

But: $-b := \int_{E^-} f < \int_{F^-} f = -a$ since $F^- \subseteq E^-$ and f only negative on E^-.

and $\int_{F^+} f < \int_{E^+} f = a$ since $F^+ \subseteq E^+$.

Hence:

$$\int_{F^+ \cup E^0 \cup E^-} f = \int_{F^+} f + \int_{E^-} f < \int_{E^+} f + \int_{E^-} f < \int_{E^+} f + \int_{F^-} f = a - a = 0$$

Hence $\int_{F^+ \cup E^0 \cup E^-} f \neq 0$, but $\mu(F^+ \cup E^0 \cup E^-) = \pi$, hence contradiction

④ $f(x) = x^2 \sin(\frac{1}{x^2})$, $g(x) = x^2 \sin(\frac{1}{x})$ for $x \neq 0$
and $f(0) = g(0) = 0$.

(i) Show f, g diff everywhere: clearly f, g differentiable for all $x \neq 0$ since they are clearly continuous for all $x \neq 0$.

Now:

$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(\frac{1}{x^2})}{x} = \lim_{x\to 0} x \sin(\frac{1}{x^2}) = 0$ by squeeze thm.

$g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x})}{x} = \lim_{x\to 0} x \sin(\frac{1}{x}) = 0$ by squeeze thm

Hence f', g' exist everywhere.

(ii) Show f is not of bdd variation on $[-1,1]$:

Consider a partition of $[-1,1]$ with $x_0 = -1$, $x_{2N+2} = 1$, and $0 \leq n \leq N$

$$x_{2n} = \frac{-1}{\sqrt{2\pi n + \frac{\pi}{2}}}, \quad x_{2n+1} = \frac{-1}{\sqrt{2\pi n + \frac{3\pi}{2}}}$$

Then:

$$|f(x_{2n+1}) - f(x_{2n})| = \left| \frac{1}{2\pi n + \frac{3\pi}{2}} \sin(2\pi n + \tfrac{3\pi}{2}) - \frac{1}{2\pi n + \frac{\pi}{2}} \sin(2\pi n + \tfrac{\pi}{2}) \right|$$

$$= \left| \frac{-1}{2\pi n + \frac{3\pi}{2}} + \frac{-1}{2\pi n + \frac{\pi}{2}} \right| = \left| \frac{1}{2\pi n + \frac{3\pi}{2}} + \frac{1}{2\pi n + \frac{\pi}{2}} \right|$$

$$\geq \left| \frac{2}{2\pi n + \frac{\pi}{2}} \right|$$

Hence $\sum_{P = \{x_i\}} |f(x_i) - f(x_{i-1})| \geq \sum_{n=0}^{N} |f(x_{2n+1}) - f(x_{2n})| \geq \sum_{n=0}^{N} \left| \frac{2}{2\pi n + \frac{\pi}{2}} \right| \to \infty$ as $N \to \infty$,

Hence, $\sup_{P=\{x_i\}} \left\{ \sum_P |f(x_i) - f(x_{i-1})| \right\} = \infty$.

(iii) Show g is of bdd variation on $[-1,1]$.

See that $\lim_{x\to 0} g(x) = \lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$ hence continuous everywhere,

and that $g'(x) = 2x \sin(\frac{1}{x}) + x^2 \cos(\frac{1}{x})(\frac{-1}{x^2}) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}) \leq 3$,

hence $g'(x)$ is bounded; then apply MVT:

$$|f(x_i) - f(x_{i-1})| = |f'(c)|\,|x_i - x_{i-1}| \leq M |x_i - x_{i-1}|$$

hence $\sum^n |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{n} M|x_i - x_{i-1}| = M \cdot 2$ since $\{x_i\}$ a partition of $[-1,1]$.