REAL ANALYSIS GRADUATE EXAM

FALL 2004

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \left(\int_a^b |f(x)|^n \, dx \right)^{1/n} = \sup_{x\in[a,b]} |f(x)|.$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Prove that

$$g(x) = \sum_{n=1}^{\infty} f\left(2^n x + \frac{1}{n}\right)$$

is integrable and

$$\int_{-\infty}^{\infty} g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx.$$

3. A Lebesgue integrable function f defined on the interval $[0, 4]$ has the property that $\int_E f(x)\, dx = 0$ for all measurable E with $m(E) = \pi$. Must $f = 0$ a.e.?

4. Let $f(x) = x^2 \sin(1/x^2)$ and $g(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = g(0) = 0$.
   (i) Show that f and g are each differentiable everywhere (including $x = 0$).
   (ii) Show that $f \notin BV([-1, 1])$.
   (iii) Show that $g \in BV([-1, 1])$.