Fall 2003

(1.) $E \subseteq \mathbb{R}$ Lebesgue measurable such that $x, y \in E,\ x \neq y \Rightarrow (x+y)/2 \notin E$. Prove $m(E)=0$

Consider an interval (a,b) and point $x_0 \in E \cap (a,b)$

Now consider the set $\frac{1}{2}x_0 + \frac{1}{2}((a,b)\cap E) = \frac{1}{2}(x_0 + (a,b)\cap E)$

$= \frac{1}{2}((a+x_0, b+x_0)\cap E) \subseteq (a,b)$ (since $x_0 \in (a,b)$).

Thus $m(\frac{1}{2}x_0 + \frac{1}{2}(a,b)\cap E) = \frac{1}{2} m((a+x_0, b+x_0)\cap E)$
$= \frac{1}{2} m((a,b)\cap E)$ since Leb meas translation invariant

Now, by hypothesis,
$[\frac{1}{2}x_0 + \frac{1}{2}((a,b)\cap E)] \cap [(a,b)\cap E] = \phi$, hence:

$$m((a,b)) \geq m((a,b)\cap E \ \cup\ \tfrac{1}{2}x_0 + \tfrac{1}{2}((a,b)\cap E))$$
$$= m((a,b)\cap E) + \tfrac{1}{2} m((a,b)\cap E)$$
$$= \tfrac{3}{2} m((a,b)\cap E).$$

$$\Rightarrow m((a,b)\cap E) \leq \tfrac{2}{3} m((a,b)) = \tfrac{2}{3}(b-a).$$

Lemma: For E with $m(E) > 0$, for $\alpha < 1$, $\exists I$ interval s.t. $m(E\cap I) > \alpha m(I)$.

Since $\mu(E) = \inf\{m(U) : E \subseteq U,\ U \text{ open}\}$, we can choose $U \supseteq E$ s.t. $m(E) + \varepsilon \geq m(U)$ $(*)$. Then, for $x \in E$, choose interval $x \in I \subseteq U$ s.t. $m(I) < \frac{\varepsilon}{1-\alpha}$. Then by $(*)$:

$$m(E\cap I) + \varepsilon \geq m(U \cap I) = m(I)$$ since $I \subseteq U$

Then:
$$m(E\cap I) \geq m(I) - \varepsilon > m(I) - (1-\alpha)m(I) = \alpha\, m(I)$$

$\Rightarrow$ Now let $\alpha = \frac{2}{3}$; then $(E\cap I) \leq \frac{2}{3} m(I)$, a contradiction.

Hence $m(E) = 0$

(2.) (a) Show that step functions are dense in $L^1(\mathbb{R})$

→ Let (ϕ_n) be a sequence of integrable simple functions such that $0 \le |\phi_1| \le \cdots \le |f|$, $\phi_n \to f$ ptwise.

Now, $|\phi_n - f| \le |\phi_n| + |f| \le 2|f| \in L^1$, hence we may apply DCT:

$0 = \int \lim_{n\to\infty} |\phi_n - f| = \lim_{n\to\infty} \int |\phi_n - f|$, hence for $\varepsilon > 0$, $\exists N$ s.t. $n \ge N \Rightarrow \int |\phi_n - f| < \varepsilon$.

So simple functions are dense in L^1.

→ Now, let $\phi = \sum a_j \chi_{E_j}$ be simple; then, since $\int_{E_j} |\phi| = \int_{E_j} |a_j|$, we have $|a_j|^{-1} \int_{E_j} |\phi| = \int_{E_j} d\mu = \mu(E_j)$, here $\mu(E_j) = |a_j|^{-1} \int_{E_j} |\phi| \le |a_j|^{-1} \int_{E_j} |f| < \infty$ since f is L^1.

Since $\mu(E_j) < \infty$, we can choose $A =$ finite union of intervals with $\mu(E \Delta A) < \varepsilon$, i.e. $\int |\chi_E - \chi_A| < \varepsilon$.

Since A is finite union of intervals, χ_A is a step function.

Then: $\int |\chi_A - f| = \int |\chi_A - \phi + \phi - f| \le \int |\chi_A - \phi| + \int |f - \phi| < 2\varepsilon$, hence step functions are dense.

2(b) $f \in L^1(\mu)$; Show $\lim_{h\to 0} \int |f(x+h) - f(x)|\, dx = 0$

Since step functions are dense in L^1, for $\varepsilon > 0$ $\exists \phi$ step fn. s.t. $\|f - \phi\|_{L^1} < \varepsilon$. Now see the following:

$$\lim_{h\to 0} \int |\phi(x+h) - \phi(x)|\, dx = \lim_{h\to 0} \int \Big| \sum_{j=0}^{n} c_j \chi_{(a_j,b_j)}(x+h) - \sum_{j=0}^{n} c_j \chi_{(a_j,b_j)}(x) \Big| dx$$

$$= \lim_{h\to 0} \int \Big| \sum_{j=0}^{n} c_j \left(\chi_{(a_j-h,b_j-h)}(x) - \chi_{(a_j,b_j)}(x) \right) \Big| dx$$

$$\leq \lim_{h\to 0} \int \sum_{j=0}^{n} |c_j| \, |\chi_{(a_j-h,b_j-h)}(x) - \chi_{(a_j,b_j)}(x)|\, dx$$

MCT (all summands are positive)

$$= \int \lim_{h\to 0} \sum_{j=0}^{n} |c_j| \, |\chi_{(a_j-h,b_j-h)}(x) - \chi_{(a_j,b_j)}(x)|\, dx$$

$$= \int \sum_{j=0}^{n} |c_j| (0)\, dx = 0.$$

Now:

$$\lim_{h\to 0} \int |f(x+h) - f(x)|\, dx = \lim_{h\to 0} \int |f(x+h) - \phi(x+h) + \phi(x+h) - \phi(x) + \phi(x) - f(x)|\, dx$$

$$\leq \lim_{h\to 0} \int |f(x+h) - \phi(x+h)| + |f(x) - \phi(x)| + |\phi(x) - \phi(x+h)|$$

$$< \varepsilon + \varepsilon + \lim_{h\to 0} \int |\phi(x) - \phi(x+h)| = 2\varepsilon.$$

hence $= 0$ since ε arbitrary.

③ $f:\mathbb{R}\to\mathbb{R}$ l.s.c. if $\liminf_{n\to\infty} f(x_n) \geq f(x)$ for $x_n \to x$.
Show f is measurable:

Recall that f is L.S.C. at x if for $\varepsilon>0$, $\exists\,\delta>0$
s.t. $|y-x|<\delta \Rightarrow f(x)-\varepsilon \leq f(y)$.
Choose $x\in f^{-1}([a,\infty))$.
For $\varepsilon>0$, $\exists\,\delta>0$ s.t. $|y-x|<\delta \Rightarrow f(x)-\varepsilon \leq f(y)$,
i.e. $y\in(x-\delta,x+\delta) \Rightarrow a-\varepsilon \leq f(x)-\varepsilon \leq f(y)$.
now, letting $\varepsilon\to 0$, $\exists\,\delta'$ s.t. $y\in(x-\delta',x+\delta') \Rightarrow a\leq f(y)$,
here there is an open interval $(x-\delta',x+\delta') \subseteq f^{-1}([a,\infty))$
containing x, hence since x was arbitrary,
the set $f^{-1}([a,\infty))$ is open, hence Borel, hence
f is measurable.

④ Show for $a > -1$:

$$\int_0^1 x^a (1-x)^{-1} \ln(x)\, dx = -\sum_{k=1}^{\infty} (a+k)^{-2}$$

$$\int_0^1 x^a \ln(x) \frac{1}{1-x}\, dx = \int_0^1 x^a \ln(x) \sum_{k=0}^{\infty} x^k \, dx$$

By MCT since all summands are positive.

$$= \sum_{k=0}^{\infty} \int_0^1 x^{k+a} \ln(x)\, dx$$

Now see the following:

$$\int_0^1 x^{k+a} \ln(x)\, dx = \left[\frac{\ln(x)\, x^{k+a+1}}{k+a+1} - \int \frac{x^{k+a+1}\, dx}{(k+a+1)x} \right]_0^1$$

int. by parts with $u = \ln(x)$, $dv = x^{k+a}dx$

$$= \left[\frac{\ln(x)\, x^{k+a+1}}{k+a+1} - \frac{x^{k+a+1}}{(k+a+1)^2} \right]_0^1 \quad (*)$$

Now, see that $\lim_{x \to 0} \ln(x)\, x^{k+a+1} = \lim_{x\to 0} \frac{\ln(x)}{\frac{1}{x^{k+a+1}}}$

L'Hopital

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{-(k+a+1)\frac{1}{x^{k+a+2}}} = \lim_{x\to 0} \frac{1}{-(k+a+1)\frac{1}{x^{k+a+1}}} = \lim_{x \to 0} \frac{-x^{k+a+1}}{(k+a+1)} = 0$$

hence $(*) = \left[0 - \frac{1}{(k+a+1)^2} - 0 + 0 \right] = \frac{1}{(k+a+1)^2}$.

So $\int_0^1 x^a \ln(x) \frac{1}{1-x} dx = \sum_{k=0}^{\infty} \frac{1}{(k+a+1)^2} = \underline{\sum_{k=1}^{\infty} \frac{1}{(k+a)^2}}$

5. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \int_{-\infty}^{+\infty} \frac{\sin(tx)}{1+t^2} dt$

Show f is continuous :

Consider a sequence such that $x_n \to x$.

Now :

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \int_{-\infty}^{+\infty} \frac{\sin(tx_n)}{1+t^2} dt \quad (*)$$

Now note that $\left| \frac{\sin(tx_n)}{1+t^2} \right| \leq \frac{1}{1+t^2} \in L^1$, hence we may apply DCT.

$$(*) = \int_{-\infty}^{+\infty} \lim_{n\to\infty} \frac{\sin(tx_n)}{1+t^2} dt = \int_{-\infty}^{+\infty} \frac{\sin(tx)}{1+t^2} dt$$

$$= f(x)$$

So $\lim_{n\to\infty} f(x_n) = f(x)$

hence f continuous.