FALL 2003 REAL ANALYSIS (MATH 525A) QUALIFYING EXAM
WEDNESDAY, SEPTEMBER 24, 2003

DIRECTIONS. Do **exactly four** of the following five problems. Start each problem on a *fresh* sheet of paper, and write on only one side. When you have completed the exam, be sure your name is printed on each page. You may keep this printed page.

Problem 1. Let E be a Lebesgue-measurable subset of $\mathbb{R}$ which has the property that $x \in E, y \in E, x \neq y$ implies that $(x+y)/2$ is *not* in E. Prove: E has Lebesgue measure zero.
Hint: Show that for an interval (a,b), for a fixed $x_0 \in (a,b) \cap E$,

$$\frac{1}{2}x_0 + \frac{1}{2}\left((a,b) \cap E\right)$$

is a subset of (a,b) which has half the measure of $(a,b) \cap E$ and is disjoint from $(a,b) \cap E$. Conclude that the measure of $(a,b) \cap E$ therefore does not exceed $\frac{2}{3}(b-a)$.

Problem 2.

(a) Show that the class of all step functions, i.e. those of the form $\sum_{j=1}^{n} c_j \chi_{(a_j,b_j)}$ with a_j, b_j finite, is dense in $L^1(\mu)$, where μ is Lebesgue on $\mathbb{R}$. [**Hint**: why is the corresponding statement true for simple functions?]

(b) Suppose $f \in L^1(\mu)$. Use the result in (a) to show that

$$\lim_{h \to 0} \int |f(x+h) - f(x)|\, dx = 0.$$

Problem 3. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be "lower semi-continuous" if

$$\liminf_{n \to \infty} f(x_n) \geq f(x) \quad \text{whenever } x_n \to x.$$

Prove that a lower semi-continuous function is Borel measurable.

Problem 4. Show that for $a > -1$

$$\int_0^1 x^a (1-x)^{-1} \ln x\, dx = -\sum_{k=1}^{\infty} (a+k)^{-2},$$

being careful to justify your calculations.

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \int_{-\infty}^{+\infty} \frac{\sin(tx)}{1+t^2}\, dt.$$

Prove: f is continuous on $\mathbb{R}$.