S2S/A

REAL ANALYSIS QUALIFYING EXAM
USC DEPARTMENT OF MATHEMATICS
SEPTEMBER 26, 2002

INSTRUCTIONS. Do all of the following problems, *on separate pieces of paper.*

Problem 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous on every interval $[a, b]$, and that both f and f' are in $L^1(\mathbb{R})$. Prove:

$$\int_{-\infty}^{\infty} f'(x)\, dx = 0.$$

Problem 2. Suppose $(X, \mathcal{A}, \mu)$ is a measure space and $\{f_n\}$, $\{g_n\}$ are sequences of measurable real-valued functions which converge in measure to f and g respectively.

(a) Prove: if $\mu(X) < +\infty$, then $\{f_n g_n\}$ converges in measure to fg.
(b) Show (by counterexample) that the hypothesis $\mu(X) < +\infty$ cannot be removed in part (a).

Problem 3. Prove: if $f \in L^1(0,1)$ and $a > -1$, then the integral

$$f_a(x) = \int_0^x (x-t)^a f(t)\, dt$$

exists for almost every $x \in (0,1)$, and that $f_a \in L^1(0,1)$.

Problem 4. Let μ be a measure on the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$ which is translation-invariant,

$$\mu(E + a) = \mu(E) \quad \text{for all Lebesgue-measurable sets } E \text{ and all } a \in \mathbb{R},$$

and that $\mu([0,1]) < +\infty$. Prove that μ is a multiple of Lebesgue measure.