REAL ANALYSIS QUALIFYING EXAM (MATH 525A)

FALL 2001

(1) Suppose $(X, \mathcal{M})$ is a measurable space, μ is a positive measure with $\mu(X) < \infty$, and f is strictly positive measurable function. Let $0 < \alpha < \mu(X)$.

(a) Show that

$$\inf \left\{ \int_E f \, \mathrm{d}\mu : \mu(E) \geq \right\} > 0.$$

HINT: Consider a set where f is bounded away from 0.

(b) Show that (a) can be false if we remove the assumption $\mu(X) < \infty$.

(2) Let μ be a finite positive measure on the Borel sets in a separable metric space X, and define $\operatorname{supp}(\mu) = \{x \in X : \mu(D) > 0 \text{ for every open set } D \ni x\}$ and $G = (\operatorname{supp}(\mu))^c$. Show that G is the largest open set with $\mu(G) = 0$.

(3) For a positive measure μ on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$, the *characteristic function* of μ is defined by

$$\phi(t) = \int \exp(itx) \, \mathrm{d}\mu(x), \qquad t \in \mathbb{R}.$$

(a) Suppose $\int |x| \, \mathrm{d}\mu(x) < \infty$. Prove that ϕ is continuously differentiable and $\phi'(0) = i \int x \, \mathrm{d}\mu(x)$.

(b) Compute the characteristic function of μ for $\mathrm{d}\mu(x) = \frac{1}{2} e^{-|x|} \, \mathrm{d}x$.

(4) Evaluate

$$\lim_{n \to \infty} \int_0^\infty \frac{e^{-x/n}}{1 + (x - n)^2} \, \mathrm{d}x$$

and justify your answer. HINT: The answer is not 0.