REAL ANALYSIS QUALIFYING EXAM (MATH 525A)

FALL 2000

(1) Let μ be a finite Borel measure on $\mathbb{R}$ and let

$$f(x) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(y)}{|x-y|^{1/2}}$$

Here $\frac{1}{|x-y|^{1/2}}$ should be interpreted as $+\infty$ when $x = y$.

(a) Prove that f is finite a.e. with respect to Lebesgue measure on $\mathbb{R}$. HINT: Consider $[-M, M]$ in place of $\mathbb{R}$.

(b) Show that f need not be finite a.e. with respect to μ.

(2) Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{f_n\}$ is a sequence of measurable functions on X such that $\{f_n(x)\}$ is a Cauchy sequence for almost every x. Show that for each $\epsilon > 0$ there is a measurable $E \subset X$ and a finite M such that $\mu(X \backslash E) < \epsilon$ and $|f_n(x)| \leq M$ for all $x \in E$ and $n \geq 1$.

(3) Suppose $\{\mu_n\}$ is a sequence of finite measures on $(X, \mathcal{M})$ and $\mu_n \to \mu$ uniformly on $\mathcal{M}$, for some set function μ. Show that μ is countably additive. (Note: We don't *assume* μ is a measure.) HINT: For $E_1, E_2, \ldots$ disjoint and $k \geq 1$, consider $\mu(\cup_{i=\infty}^{\infty} E_i) - \sum_{i=1}^{k} \mu(E_i)$.

(4) Suppose μ_1, ν_1 are positive σ-finite measures on $(X_1, \mathcal{M}_1)$ and μ_2, ν_2 are positive σ-finite measures on $(X_2, \mathcal{M}_2)$, with $\mu_1 \ll \nu_1$ and $\mu_2 \ll \nu_2$. Show that $\mu_1 \times \mu_2 \ll \nu_1 \times \nu_2$. (Here $\ll$ denotes absolute continuity.)