Real analysis: Measures 1.

$\limsup x_n = \inf_{k\geq 1}\left(\sup_{n\geq k} x_n\right)$

$\liminf x_n = \sup_{k\geq 1}\left(\inf_{n\geq k} x_n\right)$

$\mathcal{B}_X$ - generated by open sets

$\mathcal{B}_{\mathbb{R}}$ - generated by: open int, closed int, h-open int, open rays, closed rays

Continuity from below: $E_1 \subseteq E_2 \subseteq \cdots \Rightarrow \mu(\bigcup E_j) = \lim_{j\to\infty} \mu(E_j)$

" " above: $E_1 \supseteq E_2 \supseteq \cdots$ and $\mu(E_1) < \infty \Rightarrow \mu(\bigcap E_j) = \lim_{j\to\infty} \mu(E_j)$

* Complete measure = all subsets of null sets are measurable (m is complete)

$f: X \to Y$ measurable if $f^{-1}(E)$ meas. for all meas. $E \subseteq Y$.

(f_j) seq. of $\overline{\mathbb{R}}$-valued measurable functions $\Rightarrow \sup f_j, \inf f_j, \limsup f_j, \liminf f_j$ all measurable

If μ complete: (a) f meas, $f=g$ a.e. $\Rightarrow g$ meas.

(b) f_n meas, $f_n \to f$ μ-a.e. $\Rightarrow f$ meas.

$\mu(E) = \inf\{\mu(U) : U \supseteq E, U \text{ open}\}$ & $\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ cpt}\}$

$E \Delta F = (E\setminus F) \cup (F \setminus E)$:

$E \subseteq \mathbb{R}$ meas and $\mu(E) < \infty \Rightarrow \exists A$ finite union of open int. s.t. $\mu(E \Delta A) < \varepsilon$ ($E \in \mathcal{M}_{\mathbb{R}}$

Integration:

Approx. by simple: ① f meas, $f \geq 0 \Rightarrow \exists (\phi_n)$ simple s.t. $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f$, and $\phi_n \to f$ pointwise and $\phi_n \to f$ unif on set where f bdd

simple = linear combo of characteristic fn

② f meas, $\mathbb{C}$-val. $\Rightarrow \exists (\phi_n)$ simple s.t. $0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq |f|$, $\phi_n \to f$ pt wise, $\phi_n \to f$ unif. on set f bdd

$\boxed{\text{MCT}}$ $(f_n) \subseteq L^+$ s.t. $f_j \leq f_{j+1}$ $\forall j \Longrightarrow \int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$

$\boxed{\text{FATOU}}$ $(f_n) \subseteq L^+ \Longrightarrow \int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$

$\boxed{\text{DCT}}$ $(f_n) \subseteq L^1$ s.t. $|f_n| \leq g$ a.e. $\forall n \Rightarrow \int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$ ($g \in L^1$)

Corollaries: ① $(f_n) \subseteq L^+ \Rightarrow \int \sum f_n = \sum \int f_n$

* Integrable simple functions are dense in L^1

Derivative/integral exchange: $f: X \times (a,b] \to \mathbb{C}$ and $f(\cdot, t): X \to \mathbb{C}$ integrable $\forall t \in (a,b]$

(a) If $|f(x,t)| \leq g(x) \in L^1(\mu)$ and $\lim_{t\to t_0} f(x,t) = f(x,t_0)$ $\forall x$, then

$\lim_{t\to t_0} \int_X f(x,t)\, d\mu(x) = \int_X f(x,t_0)\, d\mu(x)$ (i.e. if $f(x,\cdot)$ cont. for all x, then $F(t) = \int_X f(x,t)\, d\mu(x)$ cont.)

(b) If $\frac{\partial f}{\partial t}$ exists & $\left|\frac{\partial f}{\partial t}\right| \leq g(x) \in L^1(\mu)$ then

$$\boxed{\frac{d}{dt}\int_X f(x,t)\, d\mu(x) = \int \frac{\partial f}{\partial t}(x,t)\, d\mu(x)}$$

Modes of convergence

Convergent in measure: for each $\varepsilon > 0$, $\lim_{n\to\infty} \mu(\{|f_n(x) - f(x)| \geq \varepsilon\}) = 0$

Convergence in L^1: for each $\varepsilon > 0$, $\exists N$ s.t. $n \geq N \Rightarrow \int |f_n - f| \, d\mu < \varepsilon$

Unif. convergent: for each $\varepsilon > 0$, $\exists N$ s.t. $n \geq N \Rightarrow |f - f_n| < \varepsilon$ (unif $\Rightarrow$ ptwise)
→ for all x

- $f_n \to f$ in $L^1 \Rightarrow f_n \to f$ in measure
- $f_n \to f$ in measure $\Rightarrow \exists$ subseq f_{n_j} converging to f a.e.

FUBINI / TONELLI $(X, \mu), (Y, \nu)$ σ-finite

① Tonelli: $f \in L^+(X \times Y)$; then $\int f \, d(\mu \times \nu) = \int \left[\int f \, d\nu(y)\right] d\mu(x) = \int \left[\int f \, d\mu(x)\right] d\nu$

② Fubini: $f \in L^1(\mu \times \nu)$; same

Differentiation

$\mu \perp \nu$: If $\exists E, F$ s.t. $E \cup F = X$, $E \cap F = \emptyset$, $\mu(E) = 0$, $\nu(F) = 0$

Jordan decomp: $\nu = \nu^+ - \nu^-$ s.t. $\nu^+ \perp \nu^-$, both positive $(|\nu| = \nu^+ + \nu^-)$

$\nu \ll \mu$: $\mu(E) = 0 \Rightarrow \nu(E) = 0$ ($\{\text{null sets } \mu\} \subseteq \{\text{null sets } \nu\}$)

Theorem: $\nu \ll \mu \iff$ for each $\varepsilon > 0$, $\exists \delta > 0$ s.t. $\mu(E) < \delta \Rightarrow |\nu(E)| < \varepsilon$
(ν finite, μ positive)

$\nu(E) = \int_E f \, d\mu$ (notation: $d\nu = f \, d\mu$), then clearly $\nu \ll \mu$

Radon-Nikodym ν signed measure, μ σ-finite positive

$\exists!$ measure λ and $f \in L^1(\mu)$ st. $\boxed{d\nu = d\lambda + f \, d\mu}$ ← Lebesgue decomp
and $\lambda \perp \mu$

and $\exists \nu_a$ such that $d\nu_a = f \, d\mu$

If $\nu \ll \mu$, then $d\nu = f \, d\mu$ (i.e. $f = \frac{d\nu}{d\mu}$ exists) ← R-N derivative
(i.e. part of ν that is abs. cont w.r.t. μ can be differentiated)

Properties: ν signed, μ, λ pos., $\nu \ll \mu \ll \lambda$.

① $g \in L^1(\nu) \Rightarrow g \frac{d\nu}{d\mu} \in L^1(\nu)$ and $\int g \, d\nu = \int g \frac{d\nu}{d\mu} \, d\mu$

② $\nu \ll \lambda$ and $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}$

Lebesgue Differentiation $f \in L^1_{loc}$

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)|\, dy = 0 \ \ni \ \lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y)\, dy = f(x)$$

for every family (E_r) shrinking nicely to x.

- R-N for complex: ν complex, μ positive. $\exists \lambda \ni f \in L^1(\mu)$ s.t. $\lambda \perp \mu$ $\ni$ $d\nu = d\lambda + f d\mu$
- ν reg. signed or complex Borel meas., $d\nu = d\lambda + f dm$ L-R-N decomp.

Then: $\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$ for E_r shrinking nicely to x (a.e.)

FTOC Lebesgue: $F : [a,b] \to \mathbb{C}$ TFAE:

(a) F abs. cont. on $[a,b]$

(b) $F(x) - F(a) = \int_a^x f(t)\,dt$ for some $f \in L^1([a,b])$

(c) F diff a.e. on $[a,b]$ and $F(x) - F(a) = \int_a^x F'(t)\,dt$

Continuity

- f unif cont if for each $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ for all x, y
- f abs. cont if for each $\varepsilon > 0$, $\exists \delta > 0$ s.t. $\sum (b_j - a_j) < \delta \Rightarrow \sum |f(b_j) - f(a_j)| < \varepsilon$
- f Lipschitz cont: if $\exists M$ s.t. $|f(x) - f(y)| \le M|x-y|$

f Lipschitz $\Leftrightarrow$ f abs. cont $\ni$ derivative bdd a.e.

Chebyshev: $f \in L^p$, then for $a > 0$, $\mu\{|f| > a\} \le \left(\frac{\|f\|_p}{a}\right)^p$

Minkowski: $f, g \in L^p$: $\|f+g\|_p \le \|f\|_p + \|g\|_p$

Hölder: $\frac{1}{p} + \frac{1}{q} = 1$, f, g meas $\Rightarrow$ $\|fg\|_1 \le \|f\|_p \cdot \|g\|_q$

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$$

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}, \quad |x| < 1$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n = \{x : x \in E_n \text{ for infinitely many } n\}$$

$$\liminf E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n = \{x : x \in E_n \text{ for all but finitely many } n\}$$

de Morgan: $\left(\bigcup_{\alpha \in A} E_\alpha\right)^c = \bigcap_{\alpha \in A} E_\alpha^c$

$$\left(\bigcap_{\alpha \in A} E_\alpha\right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

semicontinuity: $x_n \to x$

u.s.c. $\limsup f(x_n) \le f(x)$

at x
- $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow f(y) \le f(x) + \varepsilon$
- $\{f(x) < a\}$ open for all $a \in \mathbb{R}$

l.s.c: • $f(x) \le \liminf f(x_n)$

at x
- $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow f(x) - \varepsilon \le f(y)$
- $\{f(x) < a\}$ open for all $a \in \mathbb{R}$

$$\limsup_{x \to a} f(x) = \inf_{\delta > 0} \left(\sup_{|x-a|<\delta} f(x) \right)$$

$$\liminf_{x \to a} f(x) = \sup_{\delta > 0} \left(\inf_{|x-a|<\delta} f(x) \right)$$

Finite intersection prop: $\{F_\alpha\}_{\alpha \in A}$; $\bigcap_{\alpha \in B} F_\alpha \neq \emptyset$ for every finite $B \subseteq A$

Prop: X cpt $\iff$ every family of closed subsets $\{F_\alpha\}$ with fin-intersxn prop has $\bigcap F_\alpha \neq \emptyset$

$f \in L^1(\mu)$: ① $\varepsilon > 0$, $\exists$ int. simple fn. $\phi = \sum a_j \chi_{E_j}$ s.t. $\|f - \phi\|_{L^1} < \varepsilon$

(I.E. Int simple functions are dense in L^1)

② μ = Leb-Stieltjes on $\mathbb{R}$: • E_j = finite unions of open int

• $\exists$ cont g w/ bdd support s.t. $\|f - g\|_{L^1} < \varepsilon$ (i.e. cont. fns. dense in Lebesgue L^1)

$$m\left(\limsup_{n\to\infty} A_n\right) \ge \limsup_{n\to\infty} m(A_n) \quad \text{(cont from below)}$$

$$m\left(\liminf_{n\to\infty} A_n\right) \le \liminf_{n\to\infty} m(A_n) \quad \text{(cont from above)}$$

$A = \{x : f_n(x) \not\to f(x)\}$; $A_{\varepsilon,N} = \{|f_n(x) - f(x)| > \varepsilon$ for some $n \ge N\}$

$$\Rightarrow A = \bigcup_{\varepsilon > 0} \bigcap_{N=1}^{\infty} A_{\varepsilon,N} = \limsup_{n\to\infty} (\ldots)$$

Egoroff: $\mu(X) < \infty$, $f_n \to f$ a.e. Then for $\varepsilon > 0$, $\exists E \subseteq X$ s.t. $\mu(E) < \varepsilon$ and $f_n \to f$ unif on $X \setminus E$.

Leb diff: $f \in L^1_{loc}$, [illegible] $\frac{1}{|B|}\int$ [illegible] $= 0$ [illegible] $= f(x)$