Algebra qualifying exam, January 2025

Justify all arguments completely. Every ring R is assumed to have a unit $1 \in R$. Given a field k, a k-algebra A is a ring which is equipped with a central ring homomorphism $k \to A$. Reference specific results whenever possible.

1. Let R be a finite ring.

(a) If $r^2 = 0$ implies $r = 0$ for any $r \in R$, show that R is commutative.

(b) Show by example that the converse is false.

2. Let n be an integer which is divisible by an odd prime. Prove that the dihedral group D_n of order $2n$ is **not** nilpotent.

3. Let R be a commutative ring. Consider collections of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ and $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ in R which satisfy $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ and $\mathfrak{q}_r \not\subseteq \mathfrak{q}_s$ at all pairs of distinct indices, and suppose

$$(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m) = (\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n).$$

Prove that $m = n$ and that, after applying a permutation $\sigma \in S_n$, we have $\mathfrak{p}_i = \mathfrak{q}_{\sigma(i)}$ at all i.

4. Recall that a $\mathbb{Z}$-modules M is called torsion free if, for each nonzero integer n and nonzero $m \in M$, the element $n \cdot m$ is nonzero. Recall also that a $\mathbb{Z}$-module M is called projective if, for any surjective module map $\pi : N_0 \to N_1$ and arbitrary module map $f : M \to N_1$, there exists a module map $\widetilde{f} : M \to N_0$ which satisfies $\pi \widetilde{f} = f$.

Prove that a finitely generated $\mathbb{Z}$-module M is projective if and only if it is torsion free.

5. Let k be a field and $A(k)$ be the group ring $k(\mathbb{Z}/p\mathbb{Z})$ at a prime p. (This ring has basis provided by the elements g in $\mathbb{Z}/p\mathbb{Z}$ and multiplication $(\sum_g a_g \cdot g)(\sum_h a_h \cdot h) = \sum_{g,h} a_g a_h \cdot gh$.) Take for granted that $A(k)$ is semisimple whenever k is of characteristic 0. Provide the Artin-Wedderburn decomposition for $A(k)$ when:

(a) $k = \mathbb{Q}$.

(b) $k = \mathbb{C}$.

6. Take $R = \mathbb{C}[x_1, \ldots, x_m]$ and $A \in M_{n \times n}(R)$. Write $A = [f_{ij}]$ for functions $f_{ij} \in R$. Prove that A is invertible if and only if, at each point $z \in \mathbb{C}^m$, the complex matrix $A_z = [f_{ij}(z)]$ is invertible. [Note: You may use the fact that A is invertible if and only if its determinant is a unit in R.]