ALGEBRA EXAM JANUARY 14, 2023

Show your work. Be as clear as possible. Do all problems.

(1) Let $R = \mathbb{C}[x, y, z]/(z^2 - xy)$.
   (a) Show that R is an integral domain.
   (b) Show that R is integrally closed (hint: identify R as an integral extension of a polynomial ring).

   Proof. (a) Note that $z^2 - xy$ is irreducible over $\mathbb{C}[x, y]$ since it is a prime element in the UFD $\mathbb{C}[x, y]$. Therefore, R is an integral domain.
   (b) Note that R is an integral extension of $\mathbb{C}[x, y]$ since $z^2 - xy \in \mathbb{C}[x, y][z]$ and $R = \mathbb{C}[x, y][z]/(z^2 - xy)$. Since $\mathbb{C}[x, y]$ is integrally closed, so is R. □

(2) Let G be a finite group and let p be the smallest prime divisor of G. Assume that a Sylow p-subgroup P of G is cyclic.
   (a) Show that $N_G(P) = C_G(P)$ (hint: what is Aut(P)?).
   (b) Show that if G is solvable, then G contains a subgroup N of index p.
   (c) Show that if N is a subgroup of index p (whether or not G is solvable), then N is normal in G.

   Proof. See SP 2017 Q3 or Fall 2022 Q3. □

(3) Let F be a field extension of $\mathbb{Q}$ with $[F : \mathbb{Q}] = 60$ and $F/\mathbb{Q}$ Galois. Prove that if F contains a 9th root of 1, then $F/\mathbb{Q}$ is solvable.

   Proof. Let ζ be a primitive 9-th root of unity. Since $\mathbb{Q}(\zeta)/\mathbb{Q}$ is the splitting field of the separable polynomial $x^9 - 1$, $\mathbb{Q}(\zeta)/\mathbb{Q}$ is Galois and $|\operatorname{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})| = 9 - 3 = 6$. Hence, $\operatorname{Gal}(F/\mathbb{Q})$ has normal subgroup of index 6 by the Galois correspondence. We have the following subnormal series

$$1 \triangleleft \operatorname{Gal}(F/\mathbb{Q}(\zeta)) \triangleleft \operatorname{Gal}(F/Q).$$

   Since a group of order 6 is either S_3 or $\mathbb{Z}/6$ and a group of order 10 is either $\mathbb{Z}/10$ or D_{10}, all of them are solvable. Hence, $\operatorname{Gal}(F/\mathbb{Q})$ is solvable. □

(4) Let R be a finite ring with 1. Show that some element of R is not the sum of nilpotent elements. Give an example to show that 1 can be a sum of nilpotent elements.

   Proof. Since R is fintie, it is Artinian. By the Artin–Wedderburn theorem, we have

$$R/J(R) \cong \prod_{i=1}^{k} \operatorname{Mat}_{n_i}(F_i)$$

   where $F_i's$ are finite fields and $k \geq 1$. Since $J(R)$ is nilpotent, an element in $R/J(R)$ is nilpotent if and only if it is nilpotent in R. Note that any nilpotent matrix has minimal polynomial x^n for some n, which implies that the trace of a nilpotent matrix is 0. Take $E_{11} \in \operatorname{Mat}_{n_1}(F_1)$. Since $Tr(E_1 1) = 1$, it cannot be written as a sum of nilpotent elements; hence, a preimage of E_{11} in R is not the sum of nilpotent elements.

   For the example, we let $R = \operatorname{Mat}_2(\mathbb{Z}/2)$, then we have

$$I_2 = E_{12} + E_{21} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

□

(5) Let F be an algebraically closed field with $A \in M_n(F)$.
   (a) Show that there exist polynomials $f(x), g(x) \in F[x]$ so that $A = f(A) + g(A)$ with $f(A)$ diagonalizable and $g(A)$ nilpotent.

(b) Assuming (a), show that if $A = S + N$ with S diagonalizable, N nilpotent and $SN = NS$, then $S = f(A)$ and $N = g(A)$ (in other words, S and N are unique).

Proof. (a) Since F is algebraically closed, all eigenvalues of A exists. Suppose $f(t) = (t - a_1)^{n_1} \cdots (t - a_k)^{n_k}$ is the characteristic polynomial of A and $a_i \neq a_j$ for any $i \neq j$. By Chinese Remainder theorem, there exists a polynomial $p(t) \in F[t]$ such that $p(t) \equiv a_i \pmod{(t - a_i)^{n_i}}$ for all i and $p(t) \equiv 0 \pmod{t}$. If v_i is an eigenvector of A with eigenvalue a_i, then we have

$$p(X)v_i = (a_i I_n + c_i(X - a_i I_n)^{n_i})v_i = a_i v_i.$$

Thus, $p(X)$ is diagonalizable by the eigenvectors of A. Note that $X - p(X) = X - a_i I_n$ on the eigenspace of A with eigenvalue a_i, which is nilpotent since $\prod_{i=1}^{k}(X - a_i I_n)^n = 0$ by the Cayley-Hamilton theorem. Therefore, $X - p(X)$ is nilpotent.

(b) Suppose $A = S + N$ with S diagonalizable, N nilpotent and $SN = NS$. Then we have

$$SA = S(S + N) = S^2 + NS = AS, \quad NA = NS + N^2 = SN + N^2 = AN.$$

Thus, $Sf(A) = f(A)S$ and $Ng(A) = g(A)N$. Since $S - f(A) = N - g(A)$ is both nilpotent and diagonalizable, it must be zero.

□

(6) Let R be a ring with 1. Let M be a noetherian (left) R-module.

(a) Show that if $f : M \to M$ is a surjective R-module homomorphism, then f is an isomorphism.

(b) Show that if $f : M \to M$ is an injective R-module homomorphism, it need not be an isomorphism.

Proof. (a) One can consider the increasing chain of submodules

$$\ker f \subseteq \ker f^2 \subseteq \ker f^3 \subseteq \cdots.$$

(b) Let $M = k[x]$ where k is a field. Consider the map $f : M \to M$ that fixes k and $x \mapsto x^2$. This is an injective module homomorphism, but not surjective.

□