ALGEBRA QUALIFYING EXAM SPRING 2022

(1) Consider the polynomial ring $\mathbb{C}[x_{ij}, 1 \leq i,j \leq n]$ as the algebra of polynomial functions on the space of $n \times n$ matrices $M_n(\mathbb{C})$. Let $\mathcal{N} \subset M_n(\mathbb{C})$ be the set of all nilpotent matrices. Introduce n polynomials $P_j \in \mathbb{C}[x_{ij}, 1 \leq i,j \leq n, 1 \leq j \leq n]$, defined by $P_j(A) = \mathrm{Tr} A^j$. Prove that a polynomial $Q \in \mathbb{C}[x_{ij}, 1 \leq i,j \leq n]$ vanishes on $\mathcal{N}$ if and only if some power of Q belongs to the ideal $(P_1, P_2, ..., P_n)$.

Proof. By Hilbert Nullstellensatz theorem, some power of Q vanishes on $(P_1, P_2, ..., P_n)$ is equivalent to Q vanishes on the common zero locus of $(P_1, P_2, ..., P_n)$. So it suffices to show that $\mathcal{N}$ is defined by $(P_1, P_2, ..., P_n)$.

If $A \in \mathbb{C}[x_{ij}]$ is nilpotent, then $A^k = 0$ for some $k \in \mathbb{Z}_{>0}$. Since A is a $n \times n$ matrix, the minimal polynomial of A has degree at most n. Therefore, $A^k = 0$ for some $0 < k \leq n$, i.e. $\mathcal{N}$ is defined $(P_1, P_2, ..., P_n)$. □

(2) Let $R \subset S$ be an integral ring extension. Prove that $a \in R$ is invertible as an element of R if and only if it is as an element of S.

Proof. If $a \in R$ is invertible in R, then obviously a is invertible in S since $R \subset S$.

Conversely, we assume $a \in R$ is invertible in S, i.e. there exists $b \in S$ such that $ab = 1$. Since S is an integral extension of R, we have b has a minimal polynomial

$$f(x) = x^n + \cdots + a_1 x + a_0 \in R[x].$$

Thus, $b^n + \cdots + a_1 b + a_0 = 0$ and $a_0 \in R$ implies $b^n + \cdots a_1 b \in R$. Since $ab = 1$, we have $b^{n-1} + \cdots + a_1 \in R$. Continue this process, we have $b \in R$. Therefore, a is invertible in R. □

(3) Let R be a commutative ring and M a finitely generated R-module.
  (a) Prove that if R is a principal ideal domain, then M is projective if and only if M is torsion free.
  (b) Answer the question: does the assertion (a) remain valid if R is assumed to be a local domain?

Proof. (a) This follows directly from the structure theorem of finite modules over a PID.

(b) Note that a projective module over a local ring is free. And an ideal of an integral domain is free if and only if it is principal. Now, let $R = k[x,y]_{(x,y)}$ and $M = (x,y)R$. M is finitely generate and torsion-free since R is a local domain, but M is not free since M is not principal. So the assertion in (a) fails. □

(4) Show that the center of a simple ring is a field and that the center of a semi-simple ring is a finite direct product of fields.

Proof. For the first claim, it suffices to show that $Z(R)$ is a division ring. Let $x \in Z(R)$. Then xR is a two sided ideal since for any $r \in R$, we have $rx = xr \in xR$. Since R is simple, $xR = R$. Hence, $Z(R)$ is a field.

For the second claim, it follows directly from the Artin-Wedderburn theorem. □

(5) Set $n = |SL_2(\mathbb{F}_7)|$. For each $p|n$, find a Sylow p-subgroup of $SL_2(\mathbb{F}_7)$.

Proof. First, $n = \frac{(7^2-1)(7^2-7)}{7-1} = (7+1)(7-1)7 = 3 * 16 * 7$. A subgroup of order 7 can be

$$\{\begin{pmatrix} 1 & k \\ & 1 \end{pmatrix} \mid k \in \mathbb{Z}/7\}.$$

A subgroup of order 3 can be

$$\langle \begin{pmatrix} 2 & \\ & 4 \end{pmatrix} \rangle.$$

A subgroup of order 16 can be

$$\langle \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle.$$

□

(6) Find the Galois group of the polynomial $x^4 - 4x^2 - 21$ (over $\mathbb{Q}$). Answer the question: is this polynomial solvable in radicals?

Proof. Note that $x^4 - 4x^2 - 21 = (x^2 - 7)(x^2 + 3) = (x - \sqrt{7})(x + \sqrt{7})(x - i\sqrt{3})(x + i\sqrt{3})$. Thus, the splitting field is $\mathbb{Q}(\sqrt{7}, i\sqrt{3})$. Note that $\mathbb{Q}(\sqrt{7})/\mathbb{Q}$ and $\mathbb{Q}(i\sqrt{3})/\mathbb{Q}$ are Galois extension since they are splitting field of separable polynomials. Since $\mathbb{Q}(\sqrt{7}) \cap \mathbb{Q}(i\sqrt{3}) = \mathbb{Q}$, we have

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{7}, i\sqrt{3})/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}(\sqrt{7})/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(i\sqrt{3})/\mathbb{Q}) \cong (\mathbb{Z}/2)^2.$$

Because any abelian group is solvable, the polynomial is solvable by radicals. □