Algebra Qualifying Exam - Spring 2019

(1) Classify all groups of order 495.

Proof. Assume G is a group of order $495 = 5 \times 11 \times 9$. By Sylow's theorem, we have $n_3 = 1$ or 55, $n_5 = 1$ or 11, $n_{11} = 1$ or 45. Note that we cannot have $n_11 = 45$ and $n_5 = 11$ at the same time; otherwise, there are 44 elements of order 5 and 450 elements of order 11.

(a) Assume $n_5 = 1$. Then P_5 is normal. Note that in a group of order 99, both Sylow 3-subgroup and Sylow 11-subgroup are normal. Thus, $G \cong P_5 \rtimes (P_3 \times P_{11})$. Since $\mathrm{Aut}(P_5) = \mathrm{Aut}(\mathbb{Z}/5)$ is a group of order 4, the only possible homomorphism $P_3 \times P_{11} \to \mathrm{Aut}(\mathbb{Z}/5)$ is the trivial map. Therefore, $G \cong \mathbb{Z}/5 \times \mathbb{Z}/9 \times \mathbb{Z}/11$ or $G \cong \mathbb{Z}/5 \times \mathbb{Z}/11 \times \mathbb{Z}/3 \times \mathbb{Z}/3$.

(b) Assume $n_{11} = 1$ and G is nonabelian. By the same argument, we see that a group of order 45 is either $\mathbb{Z}/45$ or $\mathbb{Z}/5 \times (\mathbb{Z}/3)^2$. Let $H < G$ be a subgroup of order 45.
Note that $\mathrm{Aut}(\mathbb{Z}/11) \cong \mathbb{Z}/10$ is cyclic; hence, any nontrivial homomorphism $H \to \mathrm{Aut}(\mathbb{Z}/11)$ gives the same group structure of G. One such nontrivial homomorphism $H \to \mathrm{Aut}(P_{11}) = \mathrm{Aut}(\mathbb{Z}/11)$ is given by sending a generator of $\mathbb{Z}/5 < H$ to 4, because

$$4^5 \equiv 16^2 \times 4 \equiv 5^2 \times 4 \equiv 3 \times 4 \equiv 1 \pmod{11}.$$

If $P_3 \cong \mathbb{Z}/9$ then G has a presentation

$$G \cong \langle a, b, c \mid a^5 = b^{11} = c^9 = 1, ac = ca, ab = b^4 a, bc = cb \rangle.$$

If $P_3 \cong \mathbb{Z}/3 \times \mathbb{Z}/3$, then G has a presentation

$$G \cong \langle a, b, c, d \mid a^5 = b^{11} = c^3 = d^3 = 1, ac = ca, ad = da, ab = b^4 a, bc = cb, dc = cd, bd = db \rangle.$$

□

(2) If $f(x) \in \mathbb{Q}[x]$ is irreducible and $\deg f = 5$, and has precisely 2 non-real complex roots, then if E is the splitting field of f over $\mathbb{Q}$, show that $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to S_5.

Proof. See Fall 2020 Q6. □

(3) If R is a ring and M is a Noetherian (left) R-module, then prove any surjective R-module homomorphism $\varphi : M \to M$ is an isomorphism.

Proof. Assume φ is not injective. Then $\ker \varphi \neq 0$. Since φ is surjective, we have an infinite strictly increasing chain of R-submodule of M

$$\ker \varphi \subsetneq \ker \varphi^2 \subsetneq \cdots$$

which contradicts to the assumption that M is Noetherian. Therefore, φ must be injective. □

(4) If R is an Artinian ring with no non-zero nilpotent elements, then show that R is a direct sum of division rings.

Proof. Since R is Artinian, the Jacobson radical $J(R)$ is nilpotent; every elements in $J(R)$ is a nilpotent element. Because R has no nonzero nilpotent elements, $J(R) = 0$. By Artin–Wedderburn theorem,

$$R \cong \mathrm{Mat}_{n_1}(D_1) \times \cdots \times \mathrm{Mat}_{n_k}(D_k)$$

where $n_i \in \mathbb{Z}_{>0}$ and D_i are divison rings. Because R has no nonzero nilpotent elements, we must have $n_1 = \cdots = n_k = 1$; hence R is isomorphic to $D_1 \times \cdots \times D_k$, a direct product of division rings. □

(5) Let $R \subset S$ be an extension of commutative rings such that $S - R$ is closed under multiplication. Prove that R is integrally closed in S.

Proof. Assume that R is not integrally closed in S towards contradiction. Suppose $s \in S \setminus R$ is integral over R with $f(s) = 0$ for some $f = t^n + a_{n-1}t^{n-1} \cdots + a_0 \in R[t]$. By rearranging the terms, we have

$$-a_0 = s(s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1).$$

Since $S \setminus R$ is closed under multiplication, we must have $s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1 \in R$; hence,

$$-a_1 = s(s^{n-2} + a_{n-1}s^{n-3} + \cdots + a_2) \implies s^{n-2} + a_{n-1}s^{n-3} + \cdots + a_2 \in R.$$

Continue the procedure, we have $s \in R$ which contradicts to the assumption that $s \in S \setminus R$. Thus, R must be integrally closed in S. $\square$

(6) In the ring $\mathbb{C}[x, y, \frac{1}{y}]$, show that some power of $x^9 - 3y^3 + 4$ belongs to the ideal $(x^2 - y, 2x^2 - xy^2)$.

Proof. Let $I := (x^2 - y, 2x^2 - xy^2)$. By Hilbert Nullstellensatz's Theorem, the vanishing ideal of $V(I)$ is the radical of I. Thus, to show $x^9 - 3y^3 + 4 \in \sqrt{I}$ is the same as showing $x^9 - 3y^3 + 4 = 0$ on $V(I)$.

Note that $C[x, y, \frac{1}{y}] \cong \mathbb{C}[x, y]_{(y)}$. Plug $x^2 = y$ into $2x^2 - xy^2 = 0$, we get $y(2 - xy) = 0$. Since $y \neq 0$, we have $xy = 2$; hence $x^3 = 2$. Therefore,

$$x^9 - 3y^3 + 4 = (x^3)^3 - 3x^6 + 4 = 2^3 - 3 \times 2^2 + 4 = 8 - 12 + 4 = 0.$$

$\square$