ALGEBRA QUALIFYING EXAM SPRING 2014

Work all of the problems. Justify the statements in your solutions by reference to specific results, as appropriate. Partial credit is awarded for partial solutions. The set of rational numbers is $\boldsymbol{Q}$, and set of the complex numbers is $\boldsymbol{C}$. *Hand in solutions in order of the problem numbers.*

Galois + Sylow.

1. Let L be a Galois extension of a field F with $\mathrm{Gal}(L/F) \cong D_{10}$, the dihedral group of order 10. How many subfields $F \subseteq M \subseteq L$ are there, what are their dimensions over F, and how many are Galois over F?

Sylow

2. Up to isomorphism, using direct and semi-direct products, describe the possible structures of a group of order $5 \cdot 11 \cdot 61$.

Nullst

3. Let I be a nonzero ideal of $R = \boldsymbol{C}[x_1, \ldots, x_n]$. Show that R/I is a finite dimensional algebra over $\boldsymbol{C}$ if and only if I is contained in only finitely many maximal ideals of R.

4. Let R be a commutative ring with 1, and M a noetherian R module. For N a noetherian R module show that $M \otimes_R N$ is a noetherian R module. When N is an artinian R module show that $M \otimes_R N$ is an artinian R module.

rep'n of cosets'.

5. For $n \geq 5$ show that the symmetric group S_n cannot have a subgroup H with $3 \leq [S_n : H] < n$ ($[S_n : H]$ is the index of H in S_n).

Maschke + Artin Wedd

6. Let R be the group algebra $\boldsymbol{C}[S_3]$. How many nonisomorphic, irreducible, left modules does R have and why?

solvable in radicals

7. Let each of $g_1(x), g_2(x), \ldots, g_n(x) \in \boldsymbol{Q}[x]$ be irreducible of degree four and let L be a splitting field over $\boldsymbol{Q}$ for $\{g_1(x), \ldots, g_n(x)\}$. Show there is an extension field M of L that is a radical extension of $\boldsymbol{Q}$.

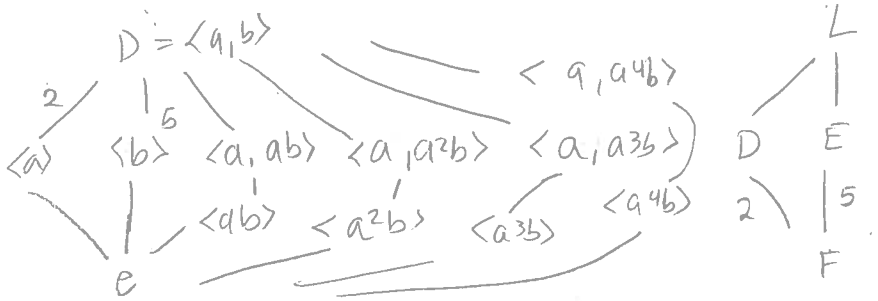
① $Gal(L/F) = D_{10}$; F field

How many subfields $F \leq M \leq L$ are there? w/ what dimensions?
How many are galois?

Pf $D_{10} = \langle a, b \mid a^5 = e, b^2 = e, bab^{-1} = a^{-1} \rangle$

subfields of $F \leq M \leq L$ are in bij. w/ subgroups of D_{10}

now, then:



where $[L:D] = 2$

since subgroups of D_{2n} have the form $\langle a^d \rangle$ w/ $d \mid n$ & $\langle a^d, a^r b \rangle$ w/ $0 \leq r < d$.

$\Rightarrow$ 10 nontrivial proper subgrps total.

Now galois ext $\xleftrightarrow{bij}$ normal subgps of D.

But by Sylow theory $\exists$ 5 sylow 2 subgroup's, so none of them are normal.

Thus $\langle a \rangle$ is normal (index 2) so only D is galois.

1) $|G| = 5 \cdot 11 \cdot 61$ Describe structure

Pf $n_{61} = 1$, so $\exists\ P \lhd G \ni |P| = 61$.

$n_{11} \equiv 1 \bmod 11$, $n_{11} = 1, 5, \cancel{61}, \cancel{5\cdot 61}$ $\quad 5 \cdot 61 \equiv 5 \cdot (-5) \equiv -25 \equiv 3 \bmod 11$

so $n_{11} = 1 \Rightarrow \exists\ Q \lhd G \ni |Q| = 11$.

Thus $PQ \cong P \times Q \lhd G$.

Now let M be a sylow 5 subgp. Then let $\varphi: M \longrightarrow Aut(P \times Q) \cong \mathbb{Z}_{10} \times \mathbb{Z}_{60}$

then $\varphi(a)^5 = 1 \Rightarrow (0, 2, 4, 6, 8) \times (12, 24, 36, 48, 0)$

so then let $\phi: P \times Q \to P \times Q$
$(c, d) \mapsto (c^i, b^j)$ w/ $i^5 \equiv 1 \bmod 11$ $j^5 \equiv 1 \bmod 61$

then $i = 4$, then $4^5 = (2^5)^2 \equiv (-1)^2 \equiv 1 \bmod 11$ (then $i^k \equiv 1 \bmod 11\ \forall k$)

and $j = \beta$ some β.

Then $G = \langle a, b, c \mid a^5 = b^{11} = c^{61} = e,\ aba^{-1} = b^{4^i},\ aca^{-1} = c^{j^k},\ bcb^{-1} = c \rangle$

(3) let I be nonzero ideal of $R = \mathbb{C}[x_1 \ldots x_n]$. Show R/I fin dim alg over $\mathbb{C}$ iff $I \subseteq$ finitely many max ideals of R.

Pf $R = \mathbb{C}[x_1 \ldots x_n]$ is Noetherian & UFD.

($\Rightarrow$) R/I has fin. dim as an alg over $\mathbb{C}$ then R/I is artinian and commutative. Thus R/I has finitely many max ideals

By the correspondence theorem $\Rightarrow \exists$ finitely many max ideals

$I \subseteq M \subseteq R \Rightarrow I \subseteq \bigcap^{k} M_\alpha$ finitely many max ideals.

$(\Leftarrow)$ Suppose $I \subseteq$ finitely many max ideals.

If $I \subseteq \bigcap^{n} M_\alpha \Rightarrow \sqrt{I} \subseteq \bigcap^{n} M_\alpha$

since $\operatorname{var}(I) \supseteq \bigcup^{k} \operatorname{var}(M_\alpha) \Rightarrow \sqrt{I} \subseteq Id(\bigcup \operatorname{var}(M_\alpha)) = \bigcap^{k} M_\alpha$.

So suppose $I = \sqrt{I}$.

Then max ideals in R have the form $\langle x - a \rangle$ for some $a \in \mathbb{C}^n$

so max ideals $\overset{bij}{\Longleftrightarrow}$ pts in $\mathbb{C}^n$.

Furthermore, $\operatorname{var}(I) = \{a_1, \ldots \} = \bigcup \{a_i\} = \bigcup \operatorname{var}(\langle x - a_i \rangle) = \operatorname{var}(\bigcap M_\alpha)$

where $I \subseteq M_\alpha$. Thus $\operatorname{var}(I) < \infty$.

But then: $R/\sqrt{I} = R/Id(\operatorname{var}(I)) = R/Id(\operatorname{var}(\bigcap M_\alpha)) = R/\bigcap M_\alpha$

since each $M_\alpha \cap M_\beta = \phi$ $\quad = R/\prod^{k} M_\alpha = \prod^{n} R/M_\alpha \simeq \mathbb{C}^k$ since $R/M_\alpha \simeq \mathbb{C}$

by the Chinese remainder thm.

Thus, $R/\sqrt{I}$ has finite dim over $\mathbb{C}$.

Now since R is noetherian then all ideals are fin. gen, so then $\sqrt{I}$ & I are both fin. gen. in particular $I \subseteq \sqrt{I}$ so that $\sqrt{I}/I$ has finite dim over $\mathbb{C} \Rightarrow R/\sqrt{I}/\sqrt{I}/I \simeq R/I$ is fin. dim over $\mathbb{C}$.

(5) $n \geq 5$, Show S_n cannot have a subgp of index $3 \leq [S_n : H] < n$

Pf Recall that for $n \geq 5$. A_n is simple and the only normal subgroups of S_n are $1, A_n, S_n$.

So if $H \leq S_n$ w/ index m $\ni$ $3 \leq m < n$ $\Rightarrow$ $H \neq S_n, A_n, \{0\}$.

then consider $H \cap A_n \triangleleft A_n$, A_n is simple so $H \cap A_n = A_n$ or $H \cap A_n = \{0\}$

if $H \cap A_n = A_n \Rightarrow A_n \leq H \leq S_n \Rightarrow H = S_n$ $\Rightarrow \Leftarrow$.

so $H \cap A_n = \{0\}$.

So consider rep'n of cosets $S_n \xrightarrow{\phi} S_m$ $\ni$ $\ker\phi \leq H$.

Then $\ker\phi \triangleleft S_n$ $\Rightarrow$ $\ker\phi = 0, S_n, A_n$.

But $A_n \not\leq H$ and $S_n \not\leq H$ $\Rightarrow$ $\ker\phi = 0$. So $S_n \hookrightarrow S_m$ w/ $m < n$.

$\Rightarrow \Leftarrow$. So H cannot exist.

(6) $R = \mathbb{C} S_3$, How many nonisomorphic, irreducible left modules does it have & why?

Pf By Mashke's theorem: $|G| = n_1^2 + n_2^2 + \dots + n_k^2$ where $k = \#$ of irreducible left modules.

Since R is semisimple (b/c $\mathbb{C}$ is alg closed) then $R = M_{n_1}(\mathbb{C}) \times \dots \times M_{n_k}(\mathbb{C})$

now $n_1 = 1 \Rightarrow 6 = 1 + n_2^2 + n_3^2 + \dots + n_k^2$

Furthermore, R cannot be abelian since S_3 is not, & $k = \#$ of conjugacy classes of $S_3 = 3$. So $6 = 1 + 1 + 4 \Rightarrow \mathbb{C}S_3 = \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$

So then R has exactly 2 nonisomorphic irreducible left modules

$\mathbb{C}$ & $M_2(\mathbb{C})$.

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(4) R- com. ring w/ 1, M noetherian R module.
N-noetherian R-module; Show $M \otimes_R N$ is noeth. R module
When N is artinian R module show that $M \otimes_R N$ is artinian R-module.

Pf If M is noetherian, then all ideals are finitely generated (in particular $M \subseteq M$)
Thus $\exists\ \phi: R^n \twoheadrightarrow M$ (since M is an ideal in itself we can consider
$(m_1) \subseteq (m_1, m_2) \subseteq \dots = (m_1, m_2, \dots, m_n) = (m_1, \dots m_{n+1})$ where $m_i \in M \setminus \{m_i\}^{k-1}$
By noetherianess this must terminate $\Rightarrow$ M is f.g.).
So then $N^n \simeq_R R^n \otimes_R N \twoheadrightarrow M \otimes_R N \Rightarrow M \otimes_R N$ is a quotient of N^n.
Thus if N is artinian/noetherian so is $N^n \Rightarrow$ so is $M \otimes_R N$.

Lemma: R- artinian + commutative $\Rightarrow \exists$ only finitely many max ideals.

Pf Suppose not. Then $\exists\ \{M_i\}_{i=1}^{\infty}$ max ideals, all distinct.
So consider $M_1 \supseteq M_1 \cdot M_2 \supsetneq \dots \Rightarrow m_1 \dots m_k = m_1 \dots m_{k+1}$ since R artinian.
Thus $m_1 + m_{k+1} = R$ since $m_1\ m_k$ are relatively prime & maximal.
So then $m_1 m_2 \dots m_k + m_2 \dots m_{k+1} = m_2 \dots m_k$; by assumption
by commutativity "
$m_1 \dots m_{k+1} = m_1 \dots m_k$ so $\Rightarrow m_1 \dots m_{k+1} + m_2 \dots m_{k+1} = m_2 \dots m_k$
& $m_1 \dots m_{k+1} \subseteq m_2 \dots m_{k+1} \Rightarrow m_2 \dots m_{k+1} = m_2 \dots m_k$
Repeating this $\Rightarrow m_k m_{k+1} = m_k \Rightarrow m_k \subsetneq m_{k+1}$ which contradicts maximality. So $\exists$ only finitely many.

(7) $g_1, \ldots, g_n \in \mathbb{Q}[x]$ irreducible of degree 4. Let L splitting field over $\mathbb{Q}$ for $\{g_i(x)\}$. Show $\exists$ ext field M of $L \ni M$ is a radical ext of $\mathbb{Q}$

Pf

$$\begin{array}{c} M \\ | \\ L \\ | \\ \mathbb{Q} \end{array}$$

, Since each g_i has deg 4 $\Rightarrow$ each g_i is solvable by radicals so that $\mathbb{Q}$(roots of g_i) is solvable by radicals, i.e. a radical extension. Let $\{\alpha_k^i\}_{k=1}^{4}$ denote roots of g_i

But then $\mathbb{Q} \subseteq \mathbb{Q}(\alpha_k^1) \subseteq \mathbb{Q}(\alpha_k^1, \alpha_k^2) \subseteq \cdots \subseteq \mathbb{Q}(\alpha_k^1, \ldots, \alpha_k^n) = L$

But each extension was solvable by radicals $\Rightarrow$ L is solvable by radicals $\Rightarrow$ L is a radical ext of $\mathbb{Q}$.

(7) $g_1, \ldots, g_n \in \mathbb{Q}[x]$ irreducible of degree 4. Let L splitting field over $\mathbb{Q}$ for $\{g_i(x)\}$. Show $\exists$ ext field M of $L \ni M$ is a radical ext of $\mathbb{Q}$

Pf $\begin{matrix} M \\ | \\ L \\ | \\ \mathbb{Q} \end{matrix}$, Since each g_i has deg 4 $\Rightarrow$ each g_i is solvable by radicals so that $\mathbb{Q}$(roots of g_i) is solvable by radicals, i.e. a radical extension. Let $\{\alpha_k^i\}_{k=1}^4$ denote roots of g_i

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Spring 2014

(5) $n \geq 5$ show S_n cannot have subgp. of order H w/

$3 \leq [S_n : H] < n$

- the only normal subgps of S_n are $1, A_n, S_n$
- A_n is simple for $n \geq 5$.

Suppose H has index m w/ $n > m \geq 3$

Then $H \cap A_n \trianglelefteq A_n \Rightarrow H \supsetneq A_n$ or $H \cap A_n = \{0\}$.

$\Downarrow$

$H = S_n \Rightarrow \Leftarrow$

note since A_n has index 2 in S_n for $n \geq 3$ then $A_n \neq H$.

Now, consider $\varphi : S_n \to S_m$ $\ni$ $\ker\varphi \leq H$ (Rep'n of cosets)

Then, $\ker\varphi \trianglelefteq S_n$ so $\ker\varphi = 1, A_n, S_n$.

By above $\ker\varphi \neq A_n \Rightarrow A_n \leq$ ~~$\ker$~~ H

if $\ker\varphi = S_n \Rightarrow H = S_n \Rightarrow$ index $H < n$.

so $\ker\varphi = 1 \Rightarrow \varphi$ is injective $\Rightarrow S_n \hookrightarrow S_m$ w/ $n < m$.

$\Rightarrow \Leftarrow$.

...

Thus such an H cannot exist.