Algebra Graduate Exam

Spring 2013

Work all the problems. Be as explicit as possible in your solutions, and justify your statements with specific reference to the results that you use. Partial credit will be given for partial solutions.

Sylow

1. Let $p > 2$ be a prime. Describe, up to isomorphism, all groups of order $2p^2$.

Zorn's Lemma

2. Let R be a commutative Noetherian ring with 1. Show that every proper ideal of R is the product of finitely many (not necessarily distinct) prime ideals of R.
(Hint: Consider the set of ideals that are not products of finitely many prime ideals. Also, note that if R is not a prime ring then IJ=(0) for some non-zero ideals I and J of R)

ask wen

3. In the polynomial ring $R = \mathbf{C}[x,y,z]$ show that there is a positive integer m, and polynomials $f, g, h \in R$ such that

$$(x^{16}y^{25}z^{81} - x^7z^{15} - yz^9 + x^5)^m = (x - y)^3f + (y - z)^5g + (x + y + z - 3)^7h.$$

nullstellensatz

ask viktor

4. Let $R \neq (0)$ be a finite ring such that for any $x \in R$ there is $y \in R$ with $xyx = x$. Show that R contains an identity element and that, for $a, b \in R$, if $ab=1$ then $ba=1$.

mg ⇒ inj b/c finite

5. Let $f(x) = x^{15} - 2$, and let L be the splitting field of f(x) over $\mathbf{Q}$. $\varphi(15)=8$

a) What is $[L:\mathbf{Q}]$?
b) Show there exists a subfield F of degree 8 that is Galois over $\mathbf{Q}$.
c) What is $\mathrm{Gal}(F/\mathbf{Q})$?
d) Show there is a subgroup of $\mathrm{Gal}(L/\mathbf{Q})$ that is isomorphic to $\mathrm{Gal}(F/\mathbf{Q})$.

galois

6. Let $F/\mathbf{Q}$ be a Galois extension of degree 60, and suppose F contains a primitive ninth root of unity. Show $\mathrm{Gal}(F/\mathbf{Q})$ is solvable.

Galois
p-groups.

7. Let n be a positive integer. Show that $f(x,y) = x^n + y^n + 1$ is irreducible in $\mathbf{C}[x,y]$. eisenstein

$(\mathbf{C}[y])[x]$

4b) if $ab = 1$ w/ $\forall x \, \exists y \ni xyx = 1$ $\overset{\text{want}}{\Longrightarrow}$ $ba = 1$

then consider $\varphi: R \to R$

$$x \mapsto xb$$

then $\varphi(a) = ab = 1$ So $\varphi(ya) = y$ for any $y \in R$

hence φ is surjective. $\Rightarrow$ injective since R is finite

So $\ker \varphi = 0$ $\Rightarrow$ $\varphi(x - xab) = \varphi(x - x) = 0$

So $\varphi(b - bab) = 0$ $\Rightarrow$ $b(1 - ab) = 0$

$(1 - ba)b = 0$

$\Rightarrow$ $\varphi(1 - ba) = 0$ since $\ker \varphi = 0$.

$\Leftrightarrow$ $1 - ba = 0$ $\Leftrightarrow$ $ba = 1$

4a) R is a finite ring w/ $xyx = x$ $(\forall x \, \exists y)$

show $1 \in R$

R - ~~ab~~ commutative $\Rightarrow$ if if $xz = 0$ for $z \neq 0$ (z ~~[illegible]~~)

$\Rightarrow$ $xyxz = xz = 0$

$\varphi: R \to R$

$a \mapsto xya$

$\varphi(x) = xyx = x$

$\Rightarrow \varphi(xa) = xyxa = xa$

$\varphi(ax) = xyax$

$\varphi(a) \cdot \varphi(x) = xya \cdot xyx$

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① $p>2$ prime. Describe all grps of order $2p^2$

Pf $n_p \equiv 1 \mod p$ & $n_p = 1$ or $\cancel{2}$ $\Rightarrow n_p = 1$ so $\exists P \trianglelefteq G$ w/ $|P| = p$.

now, let S - 2-sylow group. then consider $\varphi: \mathbb{Z}/2 \longrightarrow Aut(P)$.

CASE 1 $P = \mathbb{Z}/p \times \mathbb{Z}/p$, so $Aut(P) = GL_2(\mathbb{Z}/p)$ w/ order $(p^2-1)(p^2-p)$

so then $|Aut(P)| = (p+1)(p-1)^2 \cdot p$, so $2 \mid p+1$ & $2 \mid (p-1)$

In particular $\varphi(a)$ must have order 2, so $\varphi(a)$ must act under inversion, so if $\sigma: \underset{\langle a, b\rangle}{\mathbb{Z}/p \times \mathbb{Z}/p} \to \underset{\langle a, b\rangle}{\mathbb{Z}/p \times \mathbb{Z}/p}$; $\sigma(a,b) = (a, b)$

$= (a^{-1}, b)$

$= (a, b^{-1})$

$= (a^{-1}, b^{-1})$

for a total of 4 possibilities.

so $G = (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes_\varphi \mathbb{Z}/2$.

CASE 2 $P = \mathbb{Z}/p^2$, so $Aut(P) = (\mathbb{Z}/p^2)^\times$ w/ order $p^2 - p = p(p-1)$

Now, again φ acts by inversion so $\sigma: \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p^2$
$a \longmapsto a$ or a^{-1}.

Hence two choices for $G = P \rtimes_\varphi \mathbb{Z}/2$.

Thus we have a total of 5 choices for maps.

Now since all groups of order p^2 are abelian, we're done.

③ $R = \mathbb{C}[x,y,z]$, Show $\exists$ $m>0$ + and $f,g,h \in R$ $\ni$

$$\underbrace{(x^{16}y^{25}z^{81} - x^7z^{15} - yz^9 + x^5)}_{g(x)}{}^m = (x-y)^3 f + (y-z)^5 g + (x+y+z-3)^7 h$$

Pf W.T.S. $g(x)^m \in Id((x-y)^3) + Id((y-z)^5) + Id((x+y+z-3)^7)$

$\Rightarrow$ $g(x) \in \sqrt{\langle (x-y)^3, (y-z)^5, (x+y+z-3)^7 \rangle} = \sqrt{I}$

so if $Var(g(x)) \supseteq Var(I)$ $\Rightarrow$ $g(x) \in \sqrt{I}$ by nullstellensatz

So then $Var(I) \Rightarrow$ $x - y = 0$, $x = y$; $y - z = 0$, $y = z$ $\Rightarrow$ $x + y + z = 3$, $3z = 3$, $z = 1 \Rightarrow x, y = 1$

So $Var(I) = \{(1,1,1)\}$

So $g(1,1,1) = 1 - 1 - 1 + 1 = 0$ $\Rightarrow$ $g(x) \in \sqrt{I}$ ✓.

④ $R \neq 0$ be a finite ring $\ni$ for any $x \in R$ $\exists y$ w/ $xyx = x$

Show R contains an identity & for $a,b \in R$ if $ab = 1 \Rightarrow ba = 1$.

Pf (a) ???

(b) Suppose $1 \in R$, let $\varphi: R \to R$, $x \mapsto xb$. then $\varphi(a) = ab = 1$

so for any $r \in R$

$\varphi(ra) = rab = r$

$\Rightarrow \varphi$ is surjective.

But R is a finite ring so φ is injective.

Thus $\varphi(a) = 0$ iff $a = 0$ $\Rightarrow$ $\varphi(ab - ba) = abb - bab = b - b = 0$

$\Rightarrow$ $ab - ba = 0$ so $ab = ba$.

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② R — comm. Noetherian ring

Show $\forall$ ideal (proper) in R are the product of finitely many prime ideals

Pf Let $S = \{$ set of ideals that are not the product of finitely many prime ideals of $R \}$.

Then since R is noetherian, all chains must terminate (so by noetherian induction / Zorn's Lemma) $\exists$ a maximal element $M \in S$.

Now consider R/M. Clearly if $I \in R/M \ni I \neq 0$ then $I \notin S$

and thus I is the product of finitely many prime ideals.

If R/M is not prime $\Rightarrow \exists I, J \ni I, J \neq 0$ but $IJ = 0$ in R/M.

That is, $IJ = M$. But M is not a finite product of prime ideals

whereas I, J are $\Rightarrow\Leftarrow$. Thus R/M must be prime.

However, since R is commutative $\Rightarrow R/M$ is commutative.

Recall that a commutative ring is prime iff its zero ideal is a prime ideal.

Thus R/M prime $\Rightarrow M$ is a prime ideal. $\Rightarrow\Leftarrow$ since $M \in S$.

Thus $S = \emptyset$.

(5) $f(x) = x^{15} - 2$, L - splitting field of $f(x)$ over $\mathbb{Q}$.

(a) What is $[L : \mathbb{Q}]$

(b) Show $\exists$ subfield of degree 8 that is Galois over $\mathbb{Q}$.

(c) What is $Gal(F/\mathbb{Q})$

(d) Show $\exists$ subgrp of $Gal(L/\mathbb{Q}) \simeq Gal(F/\mathbb{Q})$

Pf (a) $f(x) = x^{15} - 2 \Rightarrow x^{15} = 2$

$x = \sqrt[15]{2}\,\omega \leftarrow$ 15th root of unity.

So there $\exists$ $\varphi(15) = \varphi(3)\cdot\varphi(5) = 2\cdot 4 = 8$ primitive roots of unity.

So then

$$\begin{array}{l} L = \mathbb{Q}(\omega, \sqrt[15]{2}) \\ \quad | \ \} \text{deg } 15 \\ F = \mathbb{Q}(\omega) \\ \quad | \ \} \text{deg } 8 \\ \mathbb{Q} \end{array} \Rightarrow [L : \mathbb{Q}] = 15 \cdot 8$$

(b) Clearly $\mathbb{Q}(\omega)$ is a subfield w/ min. poly $x^{15} - 1$

which is galois of deg $\varphi(15) = 8$

(c) $Gal(F/\mathbb{Q})$ is cyclic of degree $8 \Rightarrow Gal(F/\mathbb{Q}) = \mathbb{Z}/8$

(d) Now intermediate field are in bijection w/ subgroups of $Gal(L/\mathbb{Q})$

in particular $\exists$ subgroup of index 8, $H \leq Gal(L/\mathbb{Q})$.

Furthermore, $\omega\sqrt[15]{2}$ has order 15 since $(\omega\sqrt[15]{2})^{15} \in \mathbb{Q}$ (15 is the smallest such number) so $[\mathbb{Q}(\omega\sqrt[15]{2}) : \mathbb{Q}] = 15$.

Thus $\mathbb{Q}(\omega\sqrt[15]{2}) \longleftrightarrow$ subgroup of $Gal(L/\mathbb{Q})$ of index 15

$\Rightarrow$ it has order 8.

Since $Gal(\mathbb{Q}(\omega)/\mathbb{Q}) = Gal(F/\mathbb{Q}) \Leftrightarrow$ normal subgroup of order 15

and $Gal(L/\mathbb{Q}) \big/ Gal(L/F) \simeq Gal(F/\mathbb{Q}) \simeq \mathbb{Z}/8$

Now $\exists$ natural embedding $Gal(L/\mathbb{Q}) \big/ Gal(L/F) \hookrightarrow Gal(L/\mathbb{Q})$

$\Rightarrow H \in Gal(L/\mathbb{Q}) \ni H \simeq Gal(F/\mathbb{Q})$

(6) Let $F/\mathbb{Q}$ be galois of degree 60 w/ F containing a ninth root of unity. Show $Gal(F/\mathbb{Q})$ is solvable.

Pf if ω prim 9 root $\Rightarrow$ $\varphi(9) = \varphi(3^2) = 3^2 - 3 = 9 - 3 = 6$

so

$$\begin{array}{l} F \\ \Big| \Big\} \, 10 = 2 \cdot 5 \\ \mathbb{Q}(\omega) \\ \Big| \Big\} \, 6 \\ \mathbb{Q} \end{array}$$

Now $|Gal(\mathbb{Q}(\omega)/\mathbb{Q})| = 6 = 2 \cdot 3$.

so $n_3 = 1 \Rightarrow \exists$ normal Sylow 3 subgroup

so $0 \triangleleft N \triangleleft Gal(\mathbb{Q}(\omega)/\mathbb{Q})$

w/ $N \simeq \mathbb{Z}/3$. $\Rightarrow Gal(\mathbb{Q}(\omega)/\mathbb{Q})$ is solvable.

Now $Gal(F/\mathbb{Q}) \big/ Gal(F/\mathbb{Q}(\omega)) \simeq Gal(\mathbb{Q}(\omega)/\mathbb{Q})$ $(*)$

and $|Gal(F/\mathbb{Q}(\omega))| = 10 = 2 \cdot 5$ so again by Sylow has a normal 5 subgroup $\Rightarrow$ so it is also solvable.

Thus $Gal(F/\mathbb{Q}(\omega))$ & $Gal(\mathbb{Q}(\omega)/\mathbb{Q})$ are both solvable so by $(*)$ $Gal(F/\mathbb{Q})$ is solvable

7) $n > 0$, show $f(x,y) = x^n + y^n + 1$ is irreducible in $\mathbb{C}[x,y]$.

Pf consider $f(x,y) \in \mathbb{C}[x][y]$

so $f(x,y) = y^n + (x^n + 1)$.

Then over $\mathbb{C}$, x^n+1 factors as n linearly independent irreducible factors, $x^n+1 = \prod_{i=1}^{n} (x - \alpha_i)$ where $(\alpha_i)^n = -1$

Thus by eisenstein $x - \alpha_i$ is prime in $\mathbb{C}[x]$ (since irreducible)

but $(x-\alpha_i)^2 \nmid x^n+1 \Rightarrow f(x,y)$ is irreducible.