Algebra Qualifying Exam -Spring 2010

1. Let $f(x)= x^6+3 \in Q[x]$. Show that the Galois group of f is S_3.

2. (a) Let G be a group of order pqr, where p<q<r are primes. Show that G contains a normal subgroup of index p.
(b) Determine up to isomorphism all groups of order $3 \cdot 7 \cdot 13$.

3. Let R be a commutative Noetherian ring, and let I, J, and K be ideals of R. We say I is irreducible if $I = J \cap K \Rightarrow I = J$, or $I = K$.
(a) Show that every ideal of R is a finite intersection of irreducible ideals.
(b) Show that every irreducible ideal is primary. (An ideal I of R is primary if $R/I \neq 0$, and every zero-divisor in R/I is nilpotent.)

4. Let A be a finite-dimensional algebra over a field K, such that for every $a \in A$, $a^2 = a$. Show that A is a direct product (sum?) of fields. Which fields can arise?

5. Let G and H be finitely generated abelian groups such that $G \otimes_Z H = 0$. Show that G and H are finite and have relatively prime orders.

6. Let S and T be diagonalizable endomorphisms of a finite dimensional complex vector space. If S and T commute show that they are polynomials in each other.

7. What are the prime ideals of Z[x]? What are the maximal ideals? Carefully explain your answers.

Algebra Spring 2010

① Let $f(x) = x^6 + 3 \in \mathbb{Q}[x]$.

Show the Galois group of $f(x)$ is S_3.

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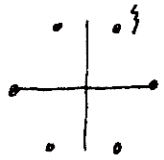
solution

Let ζ be the primative 6^{th} root of unity $\frac{1}{2} + \frac{\sqrt{3}}{2} i$.



Let $\alpha = \sqrt[6]{3}\, i$. The roots of $x^6 + 3$ are $\alpha \zeta^n$ $n = 0, 1, \ldots, 5$.

Let F be the splitting field of $f(x)$.

Then $F = \mathbb{Q}(\alpha, \zeta)$.

Since $x^6 + 3$ is irred $/\mathbb{Q}$ by Eisenstein, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 6$. we show $F = \mathbb{Q}(\alpha)$, for which we need to show $\zeta \in \mathbb{Q}(\alpha)$.

Indeed, $\alpha^3 = -\sqrt{3}\, i$, so $\zeta = \frac{1}{2} - \frac{\alpha^3}{2}$.

Thus, $[F : \mathbb{Q}] = 6$ and $|G| = 6$, where G is the Galois group of $f(x)$.

Let $\boxed{\varphi : \alpha \to \alpha\zeta}$. $\in G$.

Then $\zeta = \frac{1}{2} - \frac{\alpha^3}{2} \xrightarrow{\varphi} \frac{1}{2} - \frac{\alpha^3\zeta^3}{2} = \frac{1}{2} + \frac{\alpha^3}{2} = \frac{1}{2} - \frac{\sqrt{3}}{2}i = \zeta^5$.

$$\boxed{\zeta \xrightarrow{\varphi} \zeta^5}$$

$\varphi : \alpha \to \alpha\zeta \to \alpha\zeta\cdot\zeta^5 = \alpha \quad \Rightarrow |\varphi| = 2.$

Let $\boxed{\chi : \alpha \to \alpha\zeta^2}$.

Then $\zeta = \frac{1}{2} - \frac{\alpha^3}{2} \xrightarrow{\chi} \frac{1}{2} - \frac{\alpha^3\zeta^6}{2} = \frac{1}{2} - \frac{\alpha^3}{2} = \zeta$.

$$\boxed{\zeta \xrightarrow{\chi} \zeta}$$

$\chi : \alpha \to \alpha\zeta^2 \to \alpha\zeta^4 \to \alpha \quad \Rightarrow |\chi| = 3.$

$\varphi\chi(\alpha) = \varphi(\alpha\zeta^2) = \alpha\zeta\cdot\zeta^{10} = \alpha\zeta^5$.

$\chi\varphi(\alpha) = \chi(\alpha\zeta) = \alpha\zeta^3$. $\neq$

At this point G is nonabelian $\Rightarrow G \cong S_3$.

Indeed $\chi^2\varphi(\alpha) = \chi^2(\alpha\zeta) = \alpha\zeta^5 = \varphi\chi(\alpha)$.

$\square$

(2) (a) Let $|G| = pqr$ $p<q<r$ primes.

Show G has a normal s.g. of index p.

(b) Determine groups of order $3 \cdot 7 \cdot 13$ up to isomorphism.

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solution to (a)

Let H be a q-Sylow s.g. $\Rightarrow |H| = q$.

and let K be a r-Sylow s.g. $\Rightarrow |K| = r$.

We show either H or K is normal.

$n_r = 1$ or pq and $n_q = 1$ or r or pr.

Suppose $n_r = pq$ and $n_q = r$. Then G has at least

$1 + (r-1)pq + r(q-1) = 1 + rpq - pq + r(q-1)$ (*)

elts. $r \geq p+2$ and $q \geq p+1$. So

$pq \overset{?}{\leq} r(q-1)$, $\quad r(q-1) \geq (p+2)(q-1)$

$\quad\quad\quad

Thus HK is a subgroup of index p.

Since p is the smallest prime dividing $|G|$,

we conclude HK is normal. $\square$

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solution to (b) $\qquad |G| = 3 \cdot 7 \cdot 13$

Let P, Q, R be 3-Sylow, 7-Sylow, 13-Sylow subgroups, respectively. Then

$P = \langle x \rangle \cong \mathbb{Z}_3$, $Q = \langle y \rangle \cong \mathbb{Z}_7$, $R = \langle z \rangle \cong \mathbb{Z}_{13}$

all multiplicatively. And QR is normal.

So G is a semidirect product of P and QR whose structure is determined by group homs

$$\mathbb{Z}_3 \longrightarrow \mathrm{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_{13}) \cong \mathbb{Z}_6 \times \mathbb{Z}_{12}.$$

($QR \cong Q \times R$ since Q and R are the unique 7-Sylow and 13-Sylow subgroups in QR.)

($\mathrm{Aut}(G \times H) \cong \mathrm{Aut}G \times \mathrm{Aut}H$ when $\gcd(|G|, |H|) = 1$.)

Let α be a generator of $\mathbb{Z}_6$; we find

$\alpha : y \to y^3$ works; $y \to y^3 \to y^2 \to y^6 \to y^4 \to y^5 \to y$.

And $z \to z^2 \to z^4 \to z^8 \to z^3 \to z^6 \to z^{12} \to \cdots$ implies

$\beta : z \to z^2$ is a generator of $\mathbb{Z}_{12}$.

A group hom $\mathbb{Z}_3 \xrightarrow{\varphi} \mathbb{Z}_6 \times \mathbb{Z}_{12}$ is determined by the image of X. Since $|\varphi(X)| \mid |X|$, X is either sent to an elt of order 1 or 3.

Conversely, sending X to any such elt determines a well-defined group hom.

There are thus eight possibilities:

$$X \mapsto (1,1),\ (\alpha^2,1),\ (\alpha^4,1),$$
$$(1,\beta^4),\ (1,\beta^8),\ (\alpha^2,\beta^4),\ (\alpha^2,\beta^8),$$
$$(\alpha^4,\beta^4),\ (\alpha^4,\beta^8).$$

Each choice yields a presentation for G. e.g. (α^2,β^4)

$$\rightsquigarrow\ G_{(\alpha^2,\beta^4)} \cong \langle x,y,z : x^3 = y^7 = z^{13} = 1,\ xy\bar{x}^{-1} = y^2,\ xz\bar{x}^{-1} = z^3,\ yz = zy \rangle$$

All such groups exist by the abstract construction of the semidirect product. See attached at end.

There may be duplicates. For instance, $G_{(\alpha^2,1)} \overset{\text{not obvious}}{\cong} G_{(\alpha^4,1)}$ by the isomorphism $(x,y,z) \in G_{(\alpha^2,1)} \mapsto (x^2,y,z) \in G_{(\alpha^4,1)}$.

It is not clear to me whether there is an algorithm to check duplicates. □

(3) R comm Nöth. I, J, K ideals.

I is called irreducible if $I = J \cap K$ implies $I = J$ or $I = K$.

(a) Show every ideal I is a finite intersection of irreducible ideals.

(b) Show every irreducible ideal $I \overset{R}{\neq}$ is primary, i.e. $I \neq R$ and every zero divisor in R/I is nilpotent.

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solution to (a)

Let $\mathfrak{A}$ be the set of ideals which are not a finite intersection of irreducible ideals.

Suppose $\mathfrak{A} \neq \emptyset$. Since R is Nöth, let I be maximal in $\mathfrak{A}$. we show $I \notin \mathfrak{A}$. $\Rightarrow\Leftarrow$.

[redacted] we may choose J and K s.t. $I = J \cap K$ and

Since $I \in \mathfrak{A}$, I is not irreducible, so $I \subsetneq J$, $I \subsetneq K$.

Since I is maximal, $J = J_1 \cap \cdots \cap J_n$ and $K = K_1 \cap \cdots \cap K_m$ are intersections of finitely many irreducible ideals. But then $I = J \cap K = J_1 \cap \cdots \cap J_n \cap K_1 \cap \cdots \cap K_m$. So $\mathfrak{A} \not\ni I$. $\Rightarrow\Leftarrow$ so $\mathfrak{A} = \emptyset$. $\square$.

solution to (b)

Let $I \neq R$. Let $x \notin I$, $y \notin I$, $xy \in I$.

We want to show there is n s.t. $x^n \in I$.

For each n let $\mathrm{Ann}(x^n)$ be the ideal consisting of those $r \in R$ s.t. $rx^n \in I$.

Then $\mathrm{Ann}(x) \subset \mathrm{Ann}(x^2) \subset \dots$

Since R is Nöth, $\mathrm{Ann}(x^n) = \mathrm{Ann}(x^{n+1})$ for some n. We show $I = (I, y) \cap (I, x^n)$ from which it follows $I = (I, x^n)$, i.e. $x^n \in I$.

Let $a + ry = b + sx^n$.

Then $sx^{n+1} = ax + rxy - bx \in I$, i.e. $s \in \mathrm{Ann}(x^{n+1})$.

Since $\mathrm{Ann}(x^{n+1}) = \mathrm{Ann}(x^n)$, we conclude

$sx^n \in I$, hence $b + sx^n \in I$. $\square$

(4) A f.d. alg / field K.

Assume $a^7 = a$ for all $a \in A$.

Show A is a direct product of fields.

Which fields can arise?

Solution Assume $A \neq 0$.

Since A is f.d. alg / K

A is right (or left) Artinian.

So the Jacobson radical J consists of nilpotents.

clearly for any n, $a^n = a^d$ for $1 \leq d < 7$.

(if $n \geq 7$, then $a^n = a^{n-6}$)

so the only nilpotent is 0 : $0 = a^n = a^d \Rightarrow a = a^d a^{7-d} = 0$.

So $J = 0$. Together w/ right (or left) Artinian this gives A is *semisimple*.

So $A \cong M_{n_1}(D_1) \times \cdots \times M_{n_K}(D_K)$, D_i division rings.
$n_i = 1$ for all i since A has no nilpotents.
Every elt of D_i is a root of $x^7 - x \in D_i[x]$ which has at most 7 roots. So D_i is finite. So D_i is a finite field, each of whose elts is a root of $x^7 - x$. So $D_i \cong F_7$. Since D_i is an extension of K, K must also be F_7. □

(5) G and H f.g. abelian groups.

Assume $G \otimes_{\mathbb{Z}} H = 0$.

Show G and H are finite of relatively prime order.

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solution

$$G = \mathbb{Z}^n \oplus \bigoplus_{i,j} \mathbb{Z}_{p_i^{n_{ij}}}$$

$$H = \mathbb{Z}^m \oplus \bigoplus_{k,l} \mathbb{Z}_{q_k^{m_{kl}}}$$

From $\frac{R}{I} \otimes_R M \cong \frac{M}{IM}$ for any R-module M over a commutative ring R, it follows that $R/I \otimes_R R/J \cong R/I+J$. As a consequence, $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/\gcd(n,m)\mathbb{Z}$.

And $\oplus$, $\otimes$ commute. And $R \otimes_R M \cong M$.

So $G \otimes H \cong \mathbb{Z}^{nm} \oplus \bigoplus_{kl} \mathbb{Z}_{q_k^{m_{kl}}}^{n} \oplus \bigoplus_{ij} \mathbb{Z}_{p_i^{n_{ij}}}^{m}$

$\oplus \bigoplus_{ijkl} \mathbb{Z}_{\gcd(p_i^{n_{ij}}, q_k^{m_{kl}})}$.

First, if $G \otimes H = 0$, then $n = m = 0$.

Hence for all $ijkl$, $\mathbb{Z}_{\gcd(p_i^{n_{ij}}, q_k^{m_{kl}})}$

$= 0$ which only occurs when $\gcd(p_i^{n_{ij}}, q_k^{m_{kl}}) = 1$.

Since $|G| = \prod_{ij} p_i^{n_{ij}}$ and $|H| = \prod_{kl} q_k^{kl}$,

these groups have relatively prime orders.

□

(6) S and $T \in End(V)$ diagonalizable where V is a f.d. v.s. / $\mathbb{C}$.

If $ST = TS$, show there is $f_T, f_S \in \mathbb{C}[X]$ s.t. $T = f_T(S)$ and $S = f_S(T)$.

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solution

we must assume in addition that the algebraic multiplicities of all eigenvalues are all 1. Indeed, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ commutes w/ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and since for all $f \in \mathbb{C}[X]$, $f\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} f(1) & 0 \\ 0 & f(1) \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So we assume $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of S and $\eta_1, \ldots, \eta_n$ are distinct eigenvalues of T.

Since S and T commute they are simultaneously diagonalizable. That is, there is an invertible matrix A s.t.

$$A S A^{-1} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad \& \quad A T A^{-1} = \begin{pmatrix} \eta_1 & & \\ & \ddots & \\ & & \eta_n \end{pmatrix}.$$

Since the λ_i are distinct we may use the Chinese Remainder Theorem on the ring $\mathbb{C}[X]$ to find a polynomial f_T s.t. for all i, $f_T(X) \equiv \eta_i \pmod{(x-\lambda_i)}$.

So $f_T(\lambda_i) = \eta_i$ for all i. So

$$f_T\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \begin{pmatrix} f_T(\lambda_1) & & \\ & \ddots & \\ & & f_T(\lambda_n) \end{pmatrix} = \begin{pmatrix} \eta_1 & & \\ & \ddots & \\ & & \eta_n \end{pmatrix}$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$f_T(A S A^{-1}) = A f_T(S) A^{-1} \qquad A T A^{-1}$$

And conversely. $\square$

The harder part is showing these are the only primes.

Let $\mathfrak{p} \neq 0 \subset \mathbb{Z}[x]$ be prime. Then $\mathbb{Z} \cap \mathfrak{p}$ is prime, for $ab \in \mathbb{Z} \cap \mathfrak{p}$ implies a and b are in $\mathbb{Z}$, and one of a or b is in $\mathfrak{p}$; so a or b is in $\mathbb{Z} \cap \mathfrak{p}$.

Assume $\mathbb{Z} \cap \mathfrak{p} = \{0\}$.

Consider the extension $\mathbb{Q}[x]\mathfrak{p} \subset \mathbb{Q}[x]$.

Since $\mathbb{Z} \cap \mathfrak{p} = \{0\}$ this (extended) ideal is proper and thus is (prime and) generated over $\mathbb{Q}[x]$ by a monic irreducible polynomial $\tilde{f}(x) \in \mathbb{Q}[x]$

(here we are using that $\mathbb{Q}[x]$ is a PID).

we may take $\tilde{f}(x) = f(x) \in \mathbb{Z}[x]$ (at the cost of being monic, but we may assume primitive) by multiplying through w/ a common denominator.

Since $f(x) \in \mathbb{Z}[x] \subset \mathbb{Q}[x]$ is (primitive and) irreducible over $\mathbb{Q}[x]$ it is irreducible over $\mathbb{Z}[x]$ (this is the "trivial" direction of Gauss' lemma).

(7) Characterize the prime and maximal ideals in $\mathbb{Z}[x]$.

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solution

The answer is all primes are of the form

$\{0\}$, $(f(x))$ where $f(x) \in \mathbb{Z}[x]$ is irreducible,

(p) where $p \in \mathbb{Z}$ is prime, or

$(p, f(x))$ where $f(x) \in \mathbb{Z}_p[x]$ is irreducible

(i.e. $f(x)$ is irreducible modulo p).

Recall $\mathbb{Z}[x]$ is a U.F.D, hence

$f(x)$ irreducible implies $(f(x))$ is prime;

and if $f(x)$ is irreducible modulo p,

then $\mathbb{Z}[x]/(p, f(x)) \cong \mathbb{Z}_p[x]/(\overline{f(x)})$

is an I.D. hence $(p, f(x))$ is prime.

So all of the above are primes.

We claim $\mathfrak{p} = \mathbb{Z}[x]f(x)$. First, $f(x) \in \mathfrak{p}$ by degree considerations; indeed, $f(x) \in \mathbb{Q}[x]\mathfrak{p}$ means there is $g(x) \in \mathfrak{p}$ and $\tilde{h}(x) \in \mathbb{Q}[x]$ s.t. $f(x) = \tilde{h}(x)g(x)$, or $Nf(x) = \overbrace{h(x)}^{\in \mathbb{Z}[x]}g(x) \in \mathfrak{p}$ for some $N \geq 1$, hence $f(x) \in \mathfrak{p}$ since $\mathfrak{p}$ is prime and $\mathbb{Z} \cap \mathfrak{p} = \{0\}$.

It remains, in this case, to show $\mathfrak{p} \subset \mathbb{Z}[x]f(x)$. If $g(x) \in \mathfrak{p}$, then since $\mathfrak{p} \subset \mathbb{Q}[x]f(x)$ there is $\tilde{h}(x) \in \mathbb{Q}[x]$ s.t. $g(x) = \tilde{h}(x)f(x)$, or $Ng(x) = h(x)f(x) \in \overbrace{\mathbb{Z}[x]f(x)}^{\text{prime } \subset \mathbb{Z}[x] \text{ since } f(x) \text{ is irreducible}} \subset \mathfrak{p}$

hence $g(x) \in \mathbb{Z}[x]f(x)$ since $\mathbb{Z} \cap \mathbb{Z}[x]f(x) \subset \mathbb{Z} \cap \mathfrak{p} = \{0\}$.

This establishes $\mathfrak{p} = (f(x))$.

The previous argument is useless when $\mathbb{Z} \cap \mathfrak{p} \neq \{0\}$ since the extension $\mathbb{Q}[x]\mathfrak{p}$ equals the whole ring $\mathbb{Q}[x]$.

So we go in the other direction.

If $\mathbb{Z} \cap \mathfrak{p} \neq \{0\}$, then $\mathbb{Z} \cap \mathfrak{p} = (p)$ where $p \in \mathbb{Z}$ is prime. Let $\pi : \mathbb{Z}[x] \longrightarrow \mathbb{Z}[x]/(p) \cong \mathbb{Z}_p[x]$ be the natural projection. Then $\pi(\mathfrak{p}) \subset \mathbb{Z}_p[x]$ is prime.

Since $\mathbb{Z}_p[x]$ is a PID, the only primes are $\{0\} \in \mathbb{Z}_p[x]$ and $\mathbb{Z}_p[x]\,\tilde{f}(x)$ where $\tilde{f}(x)$ is monic & irreducible over $\mathbb{Z}_p[x]$. If $\pi(\mathfrak{p}) = \{0\}$, then $\mathfrak{p} = \pi^{-1}(\pi(\mathfrak{p})) = (p)$. ✓

Otherwise, $\pi(\mathfrak{p}) = (\tilde{f}(x))$.

If $f(x) \in \mathbb{Z}[x]$ s.t. $\pi(f(x)) = \tilde{f}(x)$, then $f(x) \in \mathfrak{p}$; pick such an $f(x)$. Then $f(x)$ is irreducible over $\mathbb{Z}[x]$, since we may choose $f(x)$ to be monic and so a nontrivial decomposition of $f(x)$ descends to a nontrivial decomposition of $\tilde{f}(x)$.

we claim $\mathfrak{p} = (p, f(x))$. we need to show the inclusion $\subseteq$. Let $g(x) \in \mathfrak{p}$.

Since $\pi(p) = \mathbb{Z}_p[x]\,\tilde{f}(x)$ we may choose $\tilde{h}(x) \in \mathbb{Z}_p[x]$ s.t. $\tilde{g}(x) = \tilde{h}(x)\tilde{f}(x)$.

Hence, $g(x) - h(x)f(x) \in (p)$

where $h(x)$ is any representative of $\tilde{h}(x)$.

So $g(x) \in (p, f(x))$ as desired.

This completes the proof of the classification. Note that the only fact about $\mathbb{Z}$ that was used is its prime ideals are principal and maximal (so that $\mathbb{Z}_p[x]$ is a PID). Hence the classification holds if we replace $\mathbb{Z}$ by any PID R.

□

Problem (2) continued

There is a spectacular result in Dummit and Foote, chapter 5, p. 184 which says if K is a cyclic group and H is any group, and if $\rho_1, \rho_2 : K \to \mathrm{Aut}(H)$ are two group homomorphisms (where we assume if K is infinite that ρ_1 and ρ_2 are injective), satisfying $\rho_1(K)$ and $\rho_2(K)$ are conjugate subgroups (e.g. if $\rho_1(K) = \rho_2(K)$), then the semidirect products $H \rtimes_{\rho_1} K$ and $H \rtimes_{\rho_2} K$ are <u>isomorphic</u>.

(In the proof one needs to use a lemma that if $a, m, n \in \mathbb{Z}$ s.t. $\gcd(a, m) = 1$ and $m \mid n$, then there is $a' \in \mathbb{Z}$ s.t. $a \equiv a' \pmod{m}$ and $\gcd(a, n) = 1$, hence there is $b \in \mathbb{Z}$ s.t. $ab \equiv 1 \pmod{n}$.)

Returning to the problem at hand, a consequence if this is that, if we let G_{mn} be the semidirect product associated w/ the group hom $X \mapsto (\alpha^m, \beta^n)$

for the permissible values $m = 0, 2, 4$ & $n = 0, 4, 8$, then

$$G_{20} \cong G_{40}, \quad G_{04} \cong G_{08},$$

$$G_{24} \cong G_{48}, \quad G_{28} \cong G_{44}.$$

(keep reading!)

Conclusion: $G_{00}, G_{20}, G_{04}, G_{24}, G_{28}$ are representatives of the five isomorphism classes of groups of order $3 \cdot 7 \cdot 13$.

The claim that these are mutually non-isomorphic is clear, except for possibly $G_{24} \not\cong G_{28}$.

We choose to show instead that $G_{24} \not\cong G_{44}$. Suppose there exists an isomorphism $\varphi : G_{24} \longrightarrow G_{44}$. Using that $\alpha : y \mapsto y^3$ and $\beta : z \to z^2$ we realize

$$G_{24} = \langle x, y, z : x^3 = y^7 = z^{13} = 1,\ xy = y^2x,\ xz = z^3x,\ yz = zy \rangle,$$

$$G_{44} = \langle x, y, z : x^3 = y^7 = z^{13} = 1,\ xy = y^4x,\ xz = z^3x,\ yz = zy \rangle.$$

Let $\tilde{x} = \varphi(x)$, $\tilde{y} = \varphi(y)$, $\tilde{z} = \varphi(z)$.

Then $|\tilde{x}| = 3$, $|\tilde{y}| = 7$, $|\tilde{z}| = 13$, and $\tilde{x}\tilde{y} = \tilde{y}^2\tilde{x} \in G_{44}$, $\tilde{x}\tilde{z} = \tilde{z}^3\tilde{x} \in G_{44}$, and $\tilde{y}\tilde{z} = \tilde{z}\tilde{y} \in G_{44}$. We show that no such triple $(\tilde{x}, \tilde{y}, \tilde{z})$ exists.

It follows from $zy = yz \in G_{44}$ (together w/ the fact the generic elt is naturally written $z^\alpha y^\beta x^\delta$ ✓) (these are the x, y, z generating G_{44}) that both the 7-Sylow and the 13-Sylow subgroups (in G_{44}) are normal (a fact true for all the G_{mn} for the same reason).

The 7-Sylow s.g. is $\{y^\beta : \beta = 0, 1, \ldots, 6\}$ and the 13-Sylow s.g. is $\{z^\alpha : \alpha = 0, 1, \ldots, 12\}$.

We claim if we can show no choice $(\tilde{x}, \tilde{y}, \tilde{z}) = (x^\delta, y^\beta, z^\alpha)$ works, then we are done. Indeed, note if the relation

$\tilde{x}\tilde{y} = \tilde{y}^2\tilde{x}$ holds in G_{44}, then

$(g\tilde{x}\bar{g}^{-1})(g\tilde{y}\bar{g}^{-1}) = (g\tilde{y}\bar{g}^{-1})^2(g\tilde{x}\bar{g}^{-1})$ $\forall g \in G_{44}$,

and similarly for $\tilde{x}\tilde{z} = \tilde{z}^3\tilde{x}$. Every elt of order 3 is of the form $\bar{g}^{-1}x^\gamma g$.

If $(\tilde{x}, \tilde{y}, \tilde{z}) = (\bar{g}^{-1}x^\gamma g, y^\beta, z^\alpha)$

satisfied the required relations, then by above $(x^\gamma, gy^\beta\bar{g}^{-1}, gz^\alpha\bar{g}^{-1}) =$

$= (x^\gamma, y^{\beta'}, z^{\alpha'})$ (by normality) would satisfy the relations, which we assumed is false, thus proving the claim.

Hence we have reduced to showing no triple of the form $(\tilde{x}, \tilde{y}, \tilde{z}) = (x^\gamma, y^\beta, z^\alpha)$ satisfies the relations, and for this we must calculate. First we find the pairs $(\tilde{x}, \tilde{y}) = (x^\delta, y^\beta)$ satisfying $\tilde{x}\tilde{y} = \tilde{y}^2\tilde{x}$ using $x^3 = 1$, $y^7 = 1$,

and $xy = y^4x$. $(\gamma, \beta) =$:

$(1,2)$ / $xy^2 \stackrel{?}{=} y^4x$ ✗ $\quad$ $(1,3)$ / $xy^3 \stackrel{?}{=} y^6x$ ✗
$y^8x = yx$ $\quad$ $y^{12}x = y^5x$

$(1,4)$ / $xy^4 \stackrel{?}{=} y^8x = yx$ ✗ $\quad$ $(1,5)$ / $xy^5 \stackrel{?}{=} y^{10}x = y^3x$ ✗
$y^{16}x = y^2x$ $\quad$ $y^{20}x = y^6x$

$(1,6)$ / $xy^6 \stackrel{?}{=} y^{12}x = y^5x$ ✗ $\quad$ $(2,1)$ / $x^2y \stackrel{?}{=} yx^2$ ✓
$y^{24}x = y^3x$ $\quad$ $y^{16}x^2 = yx^2$

$(2,2)$ / $x^2y^2 \stackrel{?}{=} y^4x^2$ ✓ etc. true for $(2,\beta)$. ‼
$y^{32}x^2 = (y^2)^2x^2 = y^4x^2$

So we fix $\tilde{x} = x^2$ and we check if any (z, α) satisfies the relation.

And it does! Trivially! The relation $xz = z^3x$ is assumed in both G_{24} & G_{44}.

Therefore we have the surprise that, in fact, $G_{24} \cong G_{44}$!! An isomorphism is given by $(x, y, z) \longmapsto (x^2, y, z)$.

□