ALGEBRA PH.D QUALIFYING EXAM SPRING 2007

1. For G a finite group with $|G| > 1$ and p a prime dividing the order of G, let $O_p(G) = \bigcap\{P \in Syl_p(G)\}$.
   a) Show that $O_p(G)$ is a normal subgroup of G.
   b) Show that if N is a normal subgroup of G with $|N| = p^k$, then $N \subseteq O_p(G)$.
   c) Prove that if G is solvable then for some p, $|O_p(G)| \neq 1$.

2. Let $F = GF(p^n)$ be a field of (exactly) p^n elements. Suppose that k is a positive integer dividing n, and set $B = \{ a^{p^k} + a^{p^{2k}} + \cdots + a^{p^n} \mid a \in F\}$.
   i) Show that $B \subseteq E$, a subfield of F with p^k elements.
   ii) Show that $B = E$.

3. Let $A \in M_n(\boldsymbol{Q})$ with $A^k = I_n$. If j is a positive integer with $(j, k) = 1$, show that $\mathrm{tr}(A) = \mathrm{tr}(A^j)$. (Hint: Consider $A \in M_n(\boldsymbol{Q}(\varepsilon))$ for $\varepsilon = e^{2\pi i/k}$, where $i^2 = -1$.)

4. Let R be a commutative ring with 1 and let M be a Noetherian R-module. If $f \in \mathrm{Hom}_R(M_R, M_R)$ is surjective, show that f is an automorphism of M_R.

5. Let $f, g \in \boldsymbol{C}[x, y]$ so that $(0, 0) \in \boldsymbol{C}^2$ is the only common zero of f and g. Prove that there is a positive integer m so that whenever $h \in \boldsymbol{C}[x, y]$ has no monomial of degree less than m, then $h \in f \cdot \mathbf{C}[x, y] + g \cdot \mathbf{C}[x, y]$.

6. For a fixed positive integer $n > 1$, describe all finite rings R so that $x^n = x$ for all $x \in R$.