ALGEBRA QUALIFYING EXAM SPRING 2006

Work all the problems. Be as explicit as possible in your solutions and justify your statements with specific reference to the results that you use. Partial credit will be given for partial solutions. Let $\boldsymbol{Q}$ denote the field of rational numbers, $\boldsymbol{C}$ the field of complex numbers, and $\mathbf{F}_q$ the finite field of q elements.

1. Up to isomorphism, describe the groups of order $3 \cdot 17 \cdot 19$.

2. i) For p a prime, $q = p^k$, and n a positive integer, describe a condition that guarantees that the multiplicative group $\mathbf{F}_q^* = (\mathbf{F}_q - \{0\}, \cdot)$ contains an element of order n.
   ii) Determine the cardinality of a splitting field L over $\mathbf{F}_3$ of $x^{13} - 1 \in \mathbf{F}_3[x]$, and the structure of $Gal(L/\mathbf{F}_3)$

3. Let $f(x) = (x^3 - 2)(x^3 - 3) \in \boldsymbol{Q}[x]$, $M \subseteq \boldsymbol{C}$ a splitting field over $\boldsymbol{Q}$ for $f(x)$, $G = Gal(M/\boldsymbol{Q})$, and $\omega \in \boldsymbol{C}$ a primitive cube root of unity.
   i) Show that $\omega \in M$.
   ii) Assume that $3^{1/3} \notin \boldsymbol{Q}(\omega, 2^{1/3}) \subseteq M$, and use this to find the order of G.
   iii) Describe how the elements of G act on M.
   iv) Determine the structure of G.

4. In $\boldsymbol{C}[x, y]$ show that for some integer $m \geq 1$, $(3x^2 + 10xy + 3y^2)^m \in (x + y - 2, x^2 + y^2 - 10)$, the ideal of $\boldsymbol{C}[x, y]$ generated by $x + y - 2$ and $x^2 + y^2 - 10$.

5. Let $g_1, g_2, \ldots, g_m, \ldots \in R$, a commutative Noetherian ring with 1, and let I be an ideal of R. Assume that for each i there is $k_i \geq 1$ so that $g_i^{k_i} \in I$. Show that there is a positive integer K so that $g_{i_1} g_{i_2} \cdots g_{i_K} \in I$ for any choices of $g_{i_j} \in \{g_t\}$.

6. Let S be a finite ring so that for each $x \in S$, $x^5 = x$.
   i) Show that S contains no nonzero nilpotent element.
   ii) Show that, up to isomorphism, S is a direct sum of copies of $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_5$.