Algebra Qualifying Exam January 2004

(1) Determine the number of nonisomorphic groups of order $2 \cdot 7 \cdot 17 \cdot 23$.

(2) Let G be a finite p-group, p a prime. Prove that the following are equivalent:
   (a) G does not contain a subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$;
   (b) Every abelian subgroup of G is cyclic;
   (c) G has a unique subgroup of order p.

(3) Let K be a splitting field of $x^4 - 2$ over the rational numbers $\mathbb{Q}$.
   (a) Find $[K : \mathbb{Q}]$ and describe the Galois group of $K/\mathbb{Q}$.
   (b) How many intermediate fields are normal (Galois) over $\mathbb{Q}$? Explain.

(4) Let k be a commutative and let R, S be commutative k algebras such that R is noetherian and S is a finitely generated k-algebra. Prove that $R \otimes_k S$ is a noetherian ring.

(5) Let k be a field, B a finitely generated k-algebra and let A be a k-subalgebra of B.
   (a) If M is a maximal ideal of B, prove that $M \cap A$ is a maximal ideal of A.
   (b) Give an example to show that this is false if B is not finitely generated.

(6) Let A be a 5 dimensional algebra over the field k of p elements, p a prime. Assume that for each nonzero $a \in A$, there exists $b \in A$ with $ab = e = e^2 \neq 0$. Find all such algebras up to isomorphism (note that this condition is satisfied by $M_n(F)$ for any field F and you may use this fact).

(7) Let F be a field and $F[x]$ the polynomial ring over F. Let M be a finitely generated free module over $F[x]$. Let $N_i, i = 1, 2, \ldots$ be a descending chain of $F[x]$-submodules of M. Prove that there exists a positive integer t so that for $i > t$, N_i/N_{i+1} is finite dimensional over F (note that $F[x]/I$ is finite dimensional over F for any nonzero ideal I).