

Algebra Qualifying Exam February 2003

Partial credit is given for partial solutions

1. Show that there are four non-isomorphic groups of order 18. For partial credit show that there are at most four non-isomorphic groups of order 18.

2. Let D_n be the dihedral group of order $2n$. Recall that D_n is generated by an element a of order n , an element b of order 2, and the relation $bab = a^{-1}$ holds.

i) Show that the derived group D'_n is:

$$D'_n = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{for } n \text{ odd;} \\ 2\mathbb{Z}/n\mathbb{Z} & \text{for } n \text{ even.} \end{cases}$$

(Hint: Show first that $a^2 \in D'_n$.) Conclude that D_n is solvable.

ii) Show that: D_n is nilpotent if and only if n is a power of 2. (Hint: Show that for $i > 1$, $\gamma_i(D_n) = \langle a^{2^{i-1}} \rangle$, where γ_i is the lower central series.)

3. Let $f(X) \in K[X]$ be an irreducible polynomial of degree 4 and G its Galois group. Show that G is isomorphic to one of the following groups: $\mathbb{Z}/4$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, D_4 , A_4 , or S_4 .

4. Let $f(X) = (X^3 - 2)(X^2 - 3) \in \mathbb{Q}[X]$. Find the splitting field K of f over \mathbb{Q} , the Galois group of the extension $\mathbb{Q} \subseteq K$ and give the number of subfields of K of each degree.

5. i) How many different similarity classes of 4×4 nilpotent matrices are there over a field F ?

ii) What is the characteristic polynomial of such a matrix?

iii) Prove or give a counterexample: If two such matrices have the same minimum polynomial and characteristic polynomial, they are similar.

6. Let $R = \mathbb{Q}[x, y, z]$.

i) Prove that every simple R -module M is finite dimensional over \mathbb{Q} .

ii) Show that there is no bound on the dimension of a simple module.

7. Consider a homogeneous system of m linear equations in n variables over \mathbb{Z} and assume that $d = n - m \geq 0$.

i) Show that the set of solutions is a free abelian group of rank at least d .

ii) What is the rank of the group of solutions (in terms of the coefficient matrix of the system of equations)?