ALGEBRA QUALIFYING EXAM FEBRUARY 2002

Partial credit is given for partial solutions

1. Describe all groups of order 2·31·61 up to isomorphism.

2. Let G be a finite solvable group. If $\langle e\rangle \neq N \lhd G$ and is N minimal (for any $H \lhd G$ with $H \subseteq N$ either $H = \langle e\rangle$ or $H = N$) show that $N \cong \mathbf{Z}_p{}^k = \mathbf{Z}_p \oplus \cdots \oplus \mathbf{Z}_p$, for p a prime and some $k \geq 1$.

3. For any prime p show that $x^4 + 1 \in \mathbf{F}_p[x]$ cannot be irreducible. ($\mathbf{F}_p$ is the field of p elements. Note that $p^2 \equiv 1 \pmod 8$ for any odd prime.)

4. Let $f(X) = f(x_1, ..., x_n) \in \mathbf{C}[x_1, ..., x_n] = R$ be irreducible. If $g(X), h(X) \in R$ with $g(\alpha) = h(\alpha)$ for all $\alpha \in \mathbf{C}^n$ satisfying $f(\alpha) = 0$, show that the images of $g(X)$ and $h(X)$ in $R/(f(X))$ are equal, that is $g(X) + (f(X)) = h(X) + (f(X))$.

5. Let R be a commutative ring with 1.
   i) Show that R is a Noetherian ring $\Leftrightarrow$ for each maximal ideal M of R the localization R_M at M is a Noetherian ring.
   ii) Show that R is a Noetherian $\Leftrightarrow$ every localization of the polynomial ring $R[x,y]$ at its maximal ideals is Noetherian.

6. If R is a right Artinian algebra over the algebraically closed field F show that R is algebraic over F of bounded degree. That is for some fixed $M > 0$ and any $r \in R$, there is some nonzero $f(x) \in F[x]$ depending on r so that $f(r) = 0$ and $\deg f \leq M$.