ALGEBRA QUALIFYING EXAM   MAY, 2000

Partial credit is given for partial solutions.

1. Up to isomorphism describe all groups of order 595 ($5\cdot 7\cdot 17$).

2. Let M be a finitely generated module over a PID R. If $M \otimes_R M \cong M$ determine the structure of M.

3. Let $\rho \in \mathbf{C}$ be a primitive p^{th} root of 1 for an odd prime p and set $L = \mathbf{Q}(\rho)$.
What is Gal($L/\mathbf{Q}$)? If m is the number of different positive integer divisors of $p - 1$, how many fields F satisfy $\mathbf{Q} \subseteq F \subseteq L$ and how many of these are Galois extensions of $\mathbf{Q}$? What are the Gal($F/\mathbf{Q}$)? Show that $[L: \mathbf{R} \cap L)] = 2$. Show that $N_{L/Q}(1 - \rho^j) = p$ for any $1 \leq j \leq p-1$.

4. Let R be a commutative Noetherian ring with 1 and let $\varphi: R[x_1, ..., x_n] \to R[x_1, ..., x_n]$ be a surjective ring homomorphism. Show that φ is an automorphism.

5. Let I be an ideal in $\mathbf{C}[x_1, ..., x_n]$.
   i) Show that there is $k > 0$ so that $(\sqrt{I})^k \subseteq I$.
   ii) Prove that if I is maximal then I/I^k is a finite dimensional $\mathbf{C}$-vector space for all $k \geq 0$.
   iii) Show that $\mathbf{C}[x_1, ..., x_n]/I$ is finite dimensional over $\mathbf{C} \Leftrightarrow \{\alpha \in \mathbf{C}^n \mid f(\alpha) = 0, \text{ all } f \in I\}$ is finite.

6. If R is a finite ring with 1 and $x, y \in R$ satisfy $xy = 1$, show that $yx = 1$.