ALGEBRA QUALIFYING EXAM NOVEMBER, 1999

Partial credit is given for partial solutions.

1. For p and q distinct primes show that any group of order p^2q is solvable.

2. Let G be a finite Abelian group so that whenever H and K are subgroups of G of the same order then $H \cong K$ as groups. Describe the possible structures of G. If $|G| = 2^3 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$, up to isomorphism how many possibilities are there for G?

3. Let $p_1, \ldots, p_k$ be distinct primes in $\mathbf{Z}$ and set $F = \mathbf{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_k}) \subseteq \mathbf{R}$.
   i) Show that F is a Galois extension of $\mathbf{Q}$ with $\mathrm{Gal}(F/\mathbf{Q}) \cong (\mathbf{Z}/2\mathbf{Z})^k$.
   ii) Show that $F = \mathbf{Q}(\sqrt{p_1} + \cdots + \sqrt{p_k})$.

4. For F a field and $R = F[x_1, \ldots, x_n]$ let M be a finitely generated R module. Show that there are positive integers s and t and an exact sequence of R modules $0 \to K \to R^s \to R^t \to M \to 0$.

5. If I is a nonzero ideal of $R = \mathbf{C}[x_1, \ldots, x_n]$ which is not maximal then if R/I is a domain, show that $\dim_{\mathbf{C}} R/I$ must be infinite.

6. If $R \neq \{0\}$ is a finite ring so that each $r \in R$ satisfies the polynomial $x^8 = x$, describe the possible structures of R.

Algebra
Nov '99

① p, q primes; show $|G| = p^2 q$ solvable.

Case $p > q$: $r_p \equiv 1 \pmod{p}$ & $r_p \mid q \Rightarrow r_p = 1$, ✗

hence the Sylow p-subgroup is normal; call it P.

Recall that a p-group is solvable, hence P solvable, but $P \trianglelefteq G$, hence $G/P \cong Q$, q-subgroup, which is also a p-group, hence solvable.

P & G/P solvable $\Rightarrow$ G solvable.

Case $p < q$: $r_p \equiv 1 \pmod{p}$ & $r_p \mid q \Rightarrow r_p = 1, q$.

$r_q \equiv 1 \pmod{q}$ & $r_q \mid p^2 \Rightarrow r_q = 1, ✗, p^2$

If $r_q = 1$, then proceed as before.

Suppose $r_q > 1$. Then $r_q = 1 + kq$ and $r_q = p^2$

$\Rightarrow kq = p^2 - 1 = (p-1)(p+1)$

$\Rightarrow q \mid (p-1)(p+1) \Rightarrow q \mid (p-1)$ or $q \mid (p+1)$ since q prime.

$q > p$, hence $q \nmid (p-1)$, hence $\underline{q \mid (p+1)}$.

But again since $q > p$, we have $q = p+1$, hence $\begin{cases} q = 3 \\ p = 2 \end{cases}$

hence $|G| = 12$. Since we suppose $r_3 \neq 4$, we have $4 \cdot 2 = 8$ elements order 3, leaving only 4 other elts, hence the Sylow 2-s.g. is normal, and proceed as before.

④ F field, $R = F[x_1, \dots, x_n]$, M fin-gen R-mod.

Show $\exists\, s, t > 0$ s.t. $0 \to K \to R^s \xrightarrow{\alpha} R^t \xrightarrow{\beta} M \to 0$ exact

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Let S be set of generators of M. Then let $R[S]$ be the free module on the generators.

and consider the map $\varphi : R[S] \to M$
$s \in S \mapsto s$

Then

$$0 \to \ker\varphi \to R[S] \to M \to 0 \text{ is exact.}$$

$$0 \to (\ker\alpha \hookrightarrow R^s \xrightarrow{\alpha} R^t \xrightarrow{\beta} M \to 0$$

(β) surjective

$\operatorname{im}\alpha = \ker\beta$

(5) $I \subseteq R = \mathbb{C}[x_1, \dots, x_n]$ not maximal

Then if R/I domain, show $\dim_{\mathbb{C}} R/I = \infty$

R/I domain $\Rightarrow$ I prime $\Rightarrow$ $I = \sqrt{I} = Id(Var(I))$.

$= \bigcap_{\alpha} (x_1 - \alpha_1, \dots, x_n - \alpha_n)$

↖ intersection of maximal ideals.

know $\mathbb{C}[x_1, \dots x_n]/\sqrt{I} \cong \mathbb{C}^{|Var\, I|}$

$\mathbb{C}[x_1, \dots x_n]/I$

~~$\mathbb{C}[x_1, \dots x_n]$ noetherian $\Rightarrow$ $I = \langle f_1, \dots, f_t \rangle$~~

Suppose finite-dimensional; then $|Var\, I|$ is finite, hence $I = \bigcap_{i=1}^{k} (x_1 - \alpha_1, \dots, x_n - \alpha_n)$ finite intersection of max. ideals

Written Qualifying Exam, Algebra, Nov. 1998

Directions. Partial credit in units of 1/4 is given for partial solutions.

1. Let G be a group of order $p^a q^b$, p, q distinct primes and a, b positive integers. Prove that if $q < p$ and the order of q mod p exceeds b then G is solvable.

2. Let G be a finitely generated abelian group (i.e., a finitely generated $\mathbb{Z}$-module).

a). Prove that G has no elements of order p, p a prime, if and only if $G \otimes_{\mathbb{Z}} \mathbb{Z}_P \cong \mathbb{Z}_P^r$ for some positive integer r, $\mathbb{Z}_P$ is the local ring of rational numbers with denominator prime to p.

b). Prove that G is projective if and only if there is an integer r such that $G \otimes_{\mathbb{Z}} H \cong H^r$ for all abelian groups H.

3. Let $\mathbb{F}_{p^n}$ be a finite field with p^n elements, p a prime. Recall that the norm map $N : \mathbb{F}_{p^n}$ is defined by $N(x) = \prod_{g \in Gal_{\mathbb{F}_p}\mathbb{F}_{p^n}} g(x)$ and the trace map is defined by $T(x) = \sum_{g \in Gal_{\mathbb{F}_p}\mathbb{F}_{p^n}} g(x)$. Determine the image of each of these maps, show that the kernel of the norm map is $\{x/g(x) : x \in \mathbb{F}_{p^n}^{\times}, g \in Gal_{\mathbb{F}_p}\mathbb{F}_{p^n}\}$ and that the kernel of the trace map is $\{x - g(x) : x \in \mathbb{F}_{p^n}, g \in Gal_{\mathbb{F}_p}\mathbb{F}_{p^n}\}$.

4. Let R be a subring of $\mathbb{C}[x_1, \ldots, x_n]$ containing $\mathbb{C}$ and assume that the field of quotients of R is $\mathbb{C}(x_1, \ldots, x_n)$. Show that there are polynomials $f_1, \ldots, f_x \in \mathbb{C}[x_1, \ldots, x_n]$ such that $d\mathbb{C}[x_1, \ldots, x_n] \subset R$ if and only if $d \in I = f_1\mathbb{C}[x_1, \ldots, x_n] + \cdots + f_s\mathbb{C}[x_1, \ldots, x_n]$. In addition, show that I cannot be a maximal ideal in $\mathbb{C}[x_1, \ldots, x_n]$.

5. Maschke's theorem implies that the group algebra $k[G]$ over a field k of characteristic zero is semisimple when G is a finite group. Using this fact,

a). Determine the structure of $\mathbb{C}[S_3]$, S_3 the symmetric group on three symbols.

b). An epimorphism of groups, $\phi : G \to H$, induces an epimorphism $\Phi : k[G] \to k[H]$ on the corresponding group rings over k. Prove that if k has characteristic 0 and G is finite then $k[H]$ is a ring direct summand of $k[G]$.

6. Determine the galois group of $x^4 - p$ over the rationals, p a prime, and determine all subfields of its splitting field which are normal over the rational numbers.

Fall '98 Algebra:

① $|G| = p^a q^b$, p, q primes, $a, b > 0$, $q < p$, $q^k \not\equiv 1 \pmod{p}$ for all $k \leq b$

Show solvable:

$r_p \equiv 1 \pmod{p}$ & $r_p \mid q^b \Rightarrow r_p \equiv 1, q, q^2, \ldots, q^b$

$\Rightarrow r_p \equiv 1$

$\Rightarrow$ the Sylow p-sg. is normal, call it $\underline{P}$.

Hence then

$G/\underline{P}$ is a q-group, hence solvable since all p-grps solvable.

Likewise, $\underline{P}$ a p-gp, hence solvable.

$\therefore G/P$ & P solvable $\Rightarrow G$ solvable

② A fin gen $\mathbb{Z}$-module.

(a) Prove G has no elts order $p \iff G \otimes \mathbb{Z}_p \cong \mathbb{Z}_p^r$.
for some $r > 0$. Here $\mathbb{Z}_p$ = local ring of rationals #'s whose denom is prime to p

$G \cong \mathbb{Z}^k \oplus A_p \oplus A_t$. WTS $A_p = 0$

($\Leftarrow$): $\mathbb{Z}_p^r \cong (\mathbb{Z}^k \oplus A_p \oplus A_t) \otimes \mathbb{Z}_p$

$\cong (\mathbb{Z}^k \otimes \mathbb{Z}_p) \oplus (A_p \otimes \mathbb{Z}_p) \oplus (A_t \otimes \mathbb{Z}_p)$ ← orders prime to p.

$\cong \mathbb{Z}_p^k \oplus (A_p \otimes \mathbb{Z}_p) \oplus 0$

$|A_t| = m$, $\gcd(m,p) = 1$

$t \otimes \frac{p}{q} \in A_t \otimes \mathbb{Z}_p \Rightarrow t \otimes \frac{mp}{mq}$ ← ok ✓

$\Rightarrow mt \otimes \frac{p}{mq} = 0 \otimes \frac{p}{mq} = 0$ ✓

$\cong \mathbb{Z}_p^h \oplus (A_p \otimes \mathbb{Z}_p)$

↑ nontrivial torsion $\mathbb{Z}$-module. unless $A_p = 0$.

$\mathbb{Z}/(p) \otimes \mathbb{Z}_p$

hence $A_p = 0$, and thus no elts order p.

($\Rightarrow$) similar.

③ $\mathbb{F}_{p^n}/\mathbb{F}_p$, $G = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \langle \sigma : x \mapsto x^p \rangle$

Recall $N(x) = \prod_{g \in G} g(x)$ & $T(x) = \sum_{g \in G} g(x)$.

Define the images and show that

$$\ker N = \{ x / g(x) : x \in \mathbb{F}_{p^n}^\times, g \in G \}$$
$$\ker T = \{ x - g(x) : x \in \mathbb{F}_{p^n}, g \in G \}$$

So $N(x) = \prod_{k=1}^{n} \sigma^k(x)$, $T(x) = \sum_{k=1}^{n} \sigma^k(x)$

Both have codomain k

T surjective by linearity and linear independence of characters.

• $\alpha \in E^\times$.

$N(\alpha) = \prod_{g \in \mathrm{Gal}} g(\alpha)$

Need: $h(N(\alpha)) = h\left(\prod_{g \in \mathrm{Gal}} g(\alpha)\right)$

$= \prod_{g \in \mathrm{Gal}} hg(\alpha)$ — left mult is surjective from grp to grp

$= \prod_{g \in \mathrm{Gal}} g(\alpha)$

$= N(\alpha)$, h arbitrary, so $N(\alpha)$ fixed by all members of Gal grp

$\Rightarrow \underline{N(\alpha) \in k^\times}$

(6) G proj. $\Leftrightarrow \exists r$ s.t. $G \otimes_{\mathbb{Z}} H \cong H^r$ for all abelian grps H

G module over PID, here projective $\Leftrightarrow$ free.

($\Rightarrow$) Suppose G proj; then $G \cong \mathbb{Z}^k$.

Let H be abelian; then $G \otimes_{\mathbb{Z}} H \cong \mathbb{Z}^k \otimes_{\mathbb{Z}} H$

$\cong (\mathbb{Z} \otimes_{\mathbb{Z}} H)^k$

$\cong H^k$.

↓ hence $\mathbb{Z}$-mod

($\Leftarrow$) G fin-gen abelian, hence $G \cong \mathbb{Z}^k \oplus G_t$.

Suppose $G \otimes_{\mathbb{Z}} H \cong H^r$ for all $\mathbb{Z}$-mod H.

$\rightsquigarrow (\mathbb{Z}^k \oplus G_t) \otimes H \cong (\mathbb{Z}^k \otimes_{\mathbb{Z}} H) \oplus (G_t \otimes_{\mathbb{Z}} H)$

$\cong H^k \oplus (G_t \otimes_{\mathbb{Z}} H)$

Hence $G_t \otimes H = 0$ for all H, hence $G_t = 0$,

hence G free, hence proj

④ $R \subseteq \mathbb{C}[x_1, \dots, x_n]$ subring of $\mathbb{C}$ and $\mathrm{Frac}(R) = \mathbb{C}(x_1, \dots, x_n)$.

Show that $\exists f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n]$ s.t.

$d\mathbb{C}[x_1, \dots, x_n] \subseteq R \iff d \in I = f_1\mathbb{C}[x_1, \dots, x_n] + \cdots + f_s\mathbb{C}[x_1, \dots, x_n]$.

($\Rightarrow$) Suppose we have $d \in \mathbb{C}[x_1, \dots, x_n]$ s.t. $(d) \subseteq R$.

WTS: R noetherian.

$\hookrightarrow (d) = (f_1, \dots, f_s) = (f_1) + \cdots + (f_s)$

$\rightarrow d \in (f_1) + \cdots + (f_s)$.

Suppose x s.t. $N(x) = 0$.

$$\prod_{k=1}^{n} \sigma^k(x) = \prod_{k=1}^{n} x^{p^k} = x^p x^{p^2} \cdots x^{p^n}$$

$$= x^{p+p^2+\cdots+p^n}$$

$$= x^{1+p+p^2+\cdots+p^{n-1}}$$

~~$0 = x^{1+p+\cdots+p^{n-1}}$~~

~~$\Rightarrow x^{\frac{p^n-1}{p-1}}$~~

~~$p^n - 1 = (p-1)(p^{n-1}+p^{n-2}+\cdots+p+1)$~~

~~$= x^{\frac{p^n}{p-1}} \cdot x^{\frac{1}{p-1}}$~~

$$= \left(\frac{x}{x^{p^k}}\right)^{1+\cdots+p^{n-1}}$$

$$= \frac{x^{1+\cdots+p^{n-1}}}{x^{p^k(1+\cdots+p^{n-1})}}$$

$$= 1$$

~~$p^2(1+p+p^2+p^3)$~~

~~$= p^2+p^3+1+p$~~

~~$p^3(1+p+p^2+p^3+p^4+p^5)$~~

~~$p^3+p^4+p^5+p^6+p+p^2$~~

ALGEBRA QUALIFYING EXAM MAY, 1997

Partial credit is given for partial solutions.

1. Up to isomorphism, describe all group of order 495.

2. Let $x^4 - 7 \in F[x]$ for $F \subseteq C$. If $F \subseteq M \subseteq C$ and M is a splitting field for $x^4 - 7$ over F, find Gal(M/F): when $F = Q$; when $F = Q[\sqrt{7}]$; and when $F = Q[i]$, with $i^2 = -1$.

3. Let M be a finitely generated F[x] module (F a field). If every submodule of M has a complement, describe the structure of M in terms of F[x]. (Recall that a submodule H of a module M has a complement if there is a submodule H' so that $M \cong H \oplus H'$; i.e. $H + H' = M$ and $H \cap H' = (0)$.)

4. Show that some power of $(x + y)(x^2 + y^4 - 2)$ is in the ideal of $C[x,y]$ generated by $x^3 + y^2$ and $y^3 + yx$.

5. Let R be a commutative Noetherian ring with no nonzero nilpotent element. Set $A = \{\text{ann } I \mid I \text{ is a nonzero ideal of } R\}$ and $M = \{\text{maximal elements in } A\}$. Prove that R embeds in a direct sum of finitely many domains as follows:
   a) Show that the elements of M are prime ideals in R.
   b) For $P \neq Q$ in M, show ann $Q \subseteq P$.
   c) Show that M is finite (consider sums of ann P_i for $P_i \in M$).
   d) Show that the intersection of the elements in M is zero.

6. Let R be a finite dimensional algebra over the field F. Assume that for every $r \in R$ there some $g(x) \in F[x]$, depending on r, so that $r + g(r)r^2 = 0$. Determine the structure of R.

① $|G| = 495 = 3^2 \cdot 5 \cdot 11$

Sylow $r_{11} \equiv 1 \pmod{11}$ & $r_{11} \mid 3^2 \cdot 5 \to r_{11} = 1, \cancel{3}, \cancel{3^2}, \cancel{3 \cdot 5}, 3^2 \cdot 5, \cancel{5}$

$r_5 \equiv 1 \pmod 5$ & $r_5 \mid 3^2 \cdot 11 \Rightarrow r_5 = 1, \cancel{3}, \cancel{3^2}, \cancel{11}, \cancel{3 \cdot 11}, 11$

$r_3 \equiv 1 \pmod 3$ & $r_3 \mid 5 \cdot 11 \Rightarrow r_3 = 1, 5, 11, 5 \cdot 11$

• Suppose $r_{11} = 3^2 \cdot 5$: Then there are $10 \cdot 3^2 \cdot 5$ elts order 11, hence $(3^2 \cdot 5 \cdot 11 - 3^2 \cdot 5 \cdot 10) = 3^2 \cdot 5 = 45$ elts other orders.

Suppose $r_3 = 5 \cdot 11$; Then there are $8 \cdot 5 \cdot 11 = 8 \cdot 55$ elts of order power of 3, but $8 \cdot 55 > 45$, hence $r_3 = 1$, P the Sylow 3-sg normal.

N an 11-sg and and $P \trianglelefteq G$ s/g 3-sg. $\Rightarrow NP \leq G$. order $11 \cdot 3^2$

• Suppose $r_{11} = 1$: Then N is normal, choose P sylow 3-s-g. and then $NP \leq G$ again

So we can always obtain a s.g. of index 5, so apply reprn on cosets: $\varphi : G \to S_5$ with $NP \leq \ker\varphi$.

So $\left|\frac{G}{\ker\varphi}\right| \mid 5!$ & $\left|\frac{G}{\ker\varphi}\right| \mid |G| = 3^2 \cdot 5 \cdot 11 \Rightarrow \left|\frac{G}{\ker\varphi}\right| \mid 5$

$\Rightarrow \left|\frac{G}{\ker\varphi}\right| = 5$ since nontrivial $\Rightarrow |\ker\varphi| = 3^2 \cdot 11 \Rightarrow NP = \ker\varphi$, and NP is normal. So now we have, for Sylow 5-s-g. Q, $NP \cap Q = 0$, hence $NPQ \cong G$, Hence $G \cong NP \rtimes Q$ and get hom. $\varphi : Q \longrightarrow Aut(NP)$

See that $|NP| = 3^2 \cdot 11$ & $r_{11} = 1 \pmod{11}$ & $r_1 \mid 3^2 \Rightarrow r_{11} = 1$

$r_3 = 1 \pmod 3$ & $r_3 \mid 11 \Rightarrow r_3 = 1$

$\Rightarrow NP \cong N \oplus P \cong \mathbb{Z}_{11} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ or $\mathbb{Z}_{11} \oplus \mathbb{Z}_9$

So then $\varphi: Q \to \mathrm{Aut}(NP) \cong \mathrm{Aut}(N) \oplus \mathrm{Aut}(P)$

$\cong \mathbb{Z}_{10} \oplus \mathrm{Aut}(P)$

Q: order 5

$N \cong \mathbb{Z}_{11}$

$P \cong \mathbb{Z}_9$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3$

$\mathbb{Z}_{10}$ $\quad$ $(\mathbb{Z}_4)^\times \cong \mathbb{Z}_{(3)(3-1)} \cong \mathbb{Z}_6$

Case $P = \mathbb{Z}_9$? $\varphi: Q \to \mathrm{Aut}(N) \oplus \mathrm{Aut}(P)$

$Q = \langle q \rangle$ $\quad$ $\mathrm{Aut}(N) = \langle a \rangle$ $\quad$ $\mathrm{Aut}(P) = \langle b \rangle$

If $\varphi(q)$ is order 1, then $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{11} \oplus \mathbb{Z}_9$. ✓ only order 5 elt in $\mathrm{Aut}(NP)$

If $\varphi(q)$ is order 5, then it must map to $(a^2, 1)$

Let $NP \cong \langle \alpha \rangle \oplus \langle \beta \rangle$ $(\alpha^{11} = 1, \beta^9 = 1)$

Then $\theta(\alpha, \beta) = (\alpha^3, \beta)$

here $\varphi(q)(\alpha, \beta) = (\alpha^3, \beta)$.

So $G \cong \langle q, \alpha, \beta : q^5 = \alpha^{11} = \beta^9 = 1,\ q\alpha q^{-1} = \alpha^3,\ q\beta q^{-1} = \beta \rangle$

$k^5 = 1 \pmod{11}$

$3^5 = 27 \cdot 9$

$= 5 \cdot 9 = 45$

$= 1 \pmod{11}$

So $k = 3$.

Case $P = \mathbb{Z}_3 \oplus \mathbb{Z}_3$ $\quad$ $\varphi: Q \to \mathrm{Aut}(N) \oplus \mathrm{Aut}(\mathbb{Z}_3 \oplus \mathbb{Z}_3)$

$\mathbb{Z}_{10} \cong \langle a \rangle$ $\quad$ $GL_2(\mathbb{F}_3)$

order 5 elts in $GL_2(\mathbb{F}_3)$? See that $|GL_2(\mathbb{F}_3)| = (3^2-1)(3^2-3)$

$= (3-1)(3+1)3(3-1)$

$= 2 \cdot 2^2 \cdot 3 \cdot 2 = 2^4 \cdot 3$

$\to$ no elts order 5

So we get the same order 5 elt

$\theta(\alpha, \beta, \gamma) = \theta(\alpha^3, \beta, \gamma)$

and $G \cong \langle q, \alpha, \beta, \gamma \mid q^5 = \alpha^{11} = \beta^3 = \gamma^3 = 1,\ q\alpha q^{-1} = \alpha^3,\ q\beta q^{-1} = \beta,\ q\gamma q^{-1} = \gamma \rangle$

Thus 1 : $G \cong \mathbb{Z}_{11} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$

(2) $x^4-7 \in F[x]$, $F \subseteq \mathbb{C}$, $F \subseteq M \subseteq \mathbb{C}$ w/ M/F spl. field for x^4-7.
Find $\mathrm{Gal}(M/F)$ when (a) $F=\mathbb{Q}$, (b) $F=\mathbb{Q}(\sqrt{7})$, (c) $F=\mathbb{Q}(i)$.

$f(x)=x^4-7=(x^2-\sqrt{7})(x^2+\sqrt{7})=(x-\sqrt[4]{7})(x+\sqrt[4]{7})(x-i\sqrt[4]{7})(x+i\sqrt[4]{7})$

So then $\mathbb{Q}(i,\sqrt[4]{7})$ is a splitting field for f over $\mathbb{Q}$

Consider tower

$\mathbb{Q}(i,\sqrt[4]{7})=M$
|) 4 ← since x^4-7 min poly over $\mathbb{Q}$ (since $\sqrt{7}\notin\mathbb{Q}$ & none of rts are in $\mathbb{Q}$)
$\mathbb{Q}(i)$
|) 2
$\mathbb{Q}$

(a) $F=\mathbb{Q}$

Therefore $[M:\mathbb{Q}]=8$, hence $|\mathrm{Gal}(M/\mathbb{Q})|=8$

Then consider $\sigma:\sqrt[4]{7}\mapsto i\sqrt[4]{7}$, $i\mapsto i$ ↳ $\sigma^4=id$

$\tau:\sqrt[4]{7}\mapsto\sqrt[4]{7}$, $i\mapsto i^3=-i$ ↳ $\tau^2=id$

So

$\tau\sigma\tau(\sqrt[4]{7})=\tau\sigma(\sqrt[4]{7})=\tau(i\sqrt[4]{7})=-i\sqrt[4]{7}=\sigma^3(\sqrt[4]{7})$

$\tau\sigma\tau(i)=\tau\sigma(-i)=\tau(-i)=i=\sigma^3(i)$

Hence $\tau\sigma\tau=\sigma^3$ hence.

$\mathrm{Gal}(M/\mathbb{Q})\cong\langle\sigma,\tau:\sigma^4=\tau^2=1,\ \tau\sigma\tau=\sigma^3=\sigma^{-1}\rangle$
$\cong D_8$.

(b) $F=\mathbb{Q}(\sqrt{7})$.

Tower: $\mathbb{Q}(i,\sqrt[4]{7})$
|) 2 since min poly of $\sqrt[4]{7}$ over $\mathbb{Q}(i,\sqrt{7})$ is $x^2-\sqrt{7}$
$\mathbb{Q}(i,\sqrt{7})$
|) 2
$\mathbb{Q}(\sqrt{7})$

So $|\mathrm{Gal}(M/F)|$ 4.

Now consider:

$$\sigma: \sqrt[4]{7} \mapsto -\sqrt[4]{7}, \quad i \mapsto i \qquad \tau: \sqrt[4]{7} \to \sqrt[4]{7}, \quad i \mapsto -i \quad \Big\} \text{ fix } \mathbb{Q}(\sqrt{7})$$

Then $\sigma^2 = id$, $\tau^2 = id$ and distinct; only are elt of order 2 in $\mathbb{Z}_4$, hence $\underline{Gal(M/\mathbb{Q}(\sqrt{7})) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2}$.

(c) $\underline{F = \mathbb{Q}(i)}$:

$M = \mathbb{Q}(i, \sqrt[4]{7})$) + by logic in pt (a).
|
$\mathbb{Q}(i)$

Consider $\sigma: \sqrt[4]{7} \mapsto i\sqrt[4]{7}$, $i \mapsto i$; clearly in $Gal(M/\mathbb{Q}(i))$ and order 4,

hence $\underline{Gal(M/\mathbb{Q}(i)) \cong \mathbb{Z}_4}$

(3) M fin-gen $F[x]$-module. & every submodule of M has a complement. Describe structure of M:

$F[x]$ a PID, so may use Fund Thm of mod. over PID so that we see $M \cong F[x]^k \oplus T$, free & torsion parts.

Now, $T \cong F[x]/(p_1^{e_1}) \oplus \cdots \oplus F[x]/(p_n^{e_n})$

suppose $e_i > 1$; then have is submodule $F[x]/(p_i) \subseteq F[x]/(p_i^{e_i})$ that does not have a complement.

Hence $e_i = 1$ $\forall i$, and $T \cong F[x]/(p_1) \oplus \cdots \oplus F[x]/(p_n)$.

So: $M \cong F[x]^k \oplus F[x]/(p_1) \oplus \cdots \oplus F[x]/(p_n)$

④ Show some power of $(x+y)(x^2+y^4-2)$ is in $(x^3+y^2, y^3+yx) \subseteq \mathbb{C}[x,y]$.

I

We therefore want to show that

$$(x+y)(x^2+y^4-2) \in \sqrt{(x^3+y^2, y^3+yx)} = Id(Var(x^3+y^2, y^3+yx))$$

Now, for $(x,y) \in Var(x^3+y^2, y^3+yx)$, we $\begin{cases} x^3+y^2=0 \\ y^3+yx=0 \end{cases}$

$$\rightarrow \begin{matrix} x^3 = -y^2 \\ y^3 = -yx \end{matrix} \Rightarrow \begin{matrix} x^3 = -y^2 \\ y^2 = -x \end{matrix} \Rightarrow x^3 = -x \Rightarrow x^2 = -1$$

$$\Rightarrow x = \pm i$$

$$\Rightarrow \begin{cases} y^2 = i \\ y^2 = -i \end{cases} \Rightarrow \begin{cases} y = \pm\sqrt{i} \Rightarrow \{(i, \sqrt{i}), (i, -\sqrt{i})\} \\ y = \pm i\sqrt{i} \Rightarrow \{(-i, i\sqrt{i}), (-i, -i\sqrt{i})\} \end{cases} \in Var(I)$$

See that $i^2 + (\sqrt{i})^4 = -1 + i^2 = -1 + -1 = -2$

$i^2 + (-\sqrt{i})^4 = -1 + i^2 = -2$

$(-i)^2 + (i\sqrt{i})^4 = -1 + 1 \cdot (-1) = -2$

$(-i)^2 + (-i\sqrt{i})^4 = -1 + -1 = -2$

Also, if $x=0 \Leftrightarrow y=0$, hence $(0,0) \in Var(I)$

but $(0+0) = 0$

So $(x+y)(x^2+y^4-2) = 0$ for all points in $Var(I)$,

hence $\in Id(Var(I)) \rightarrow \in \sqrt{I}$ ✓

(5) R commutative noeth ring with no nonzero nilpotent elt.

Let $A = \{\operatorname{ann} I : I \subsetneq R \text{ nonzero ideal}\}$

$\mathcal{M} = \{\text{maximal elts. in } A\}$.

Prove that R embeds into direct sum of finitely many domains

(a) Elements of $\mathcal{M}$ are prime ideals in R:

Let $M \in \mathcal{M}$. Then $M = \operatorname{ann} I$ some I.

Suppose $xy \in M = \operatorname{ann} I \Rightarrow xyI = 0 \Rightarrow x \in \operatorname{ann}(yI)$

and note that $\operatorname{ann}_M(I) \subseteq \operatorname{ann}(yI)$ (since $xI = 0 \Rightarrow xyI = 0$ since $yI \subseteq I$)

But M was maximal, so $\operatorname{ann}(I) = \operatorname{ann}(yI)$

$\Rightarrow x \in \operatorname{ann}(I) = M$

$\Rightarrow$ M prime ideal.

(b) $P \neq Q$ in M; show $\operatorname{ann} Q \subseteq P$:

$x \in \operatorname{Ann}(Q)$; $P \neq Q \Rightarrow \exists q \in Q \setminus P \Rightarrow xq = 0$

$\Rightarrow xq \in P$, but $q \notin P$,

hence $x \in P$ since P prime.

So $\operatorname{Ann}(Q) \subseteq P$.

(c) Show M is finite:

Recall $\operatorname{ann} P \subseteq \operatorname{ann} P + \operatorname{ann} Q$

Let $\{P_i\} \subseteq \mathcal{M}$; then $\operatorname{ann} P_1 \subseteq \operatorname{ann} P + \operatorname{ann} P_2 \subseteq \cdots \subseteq \sum_{i=1}^{n} \operatorname{ann} P_i$

But R noeth, hence $\exists$ n s.t.

$$\sum_{i=1}^{n} \operatorname{ann} P_i = \sum_{i=1}^{n+1} \operatorname{ann} P_i = \cdots$$

By part (b)

$\Rightarrow \operatorname{ann} P_{n+1} \subseteq \sum_{i=1}^{n} \operatorname{ann}(P_i) \subseteq P_{n+1}$, hence $\operatorname{ann} P_{n+1} \subseteq \bigcap P_i$

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Partial credit is given for partial solutions.

1. Let G be a finite group and p a prime number. Let P be a p-Sylow subgroup of G and denote the normalizer of P in G by $N_G(P)$.
   i) Show that $N_G(P) = N_G(N_G(P))$.
   ii) If K is a normal subgroup of G and K contains P, show that $G = KN_G(P)$.
   iii) If no proper subgroup of G is its own normalizer, show that the center of G is not trivial.

2. Up to isomorphism describe all finitely generated Abelian groups which satisfy all of the following properties: i) $G \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^2$; ii) $G \otimes_{\mathbb{Z}} (\mathbb{Z}/7\mathbb{Z}) \cong (\mathbb{Z}/7\mathbb{Z})^3$; and iii) for any prime $p \neq 7$, $G \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^2$.

3. Let R be a left Artinian ring with Jacobson radical J(R). If $R \neq J(R)$ show that R is a left Noetherian ring.

4. Determine if each of the following polynomials is irreducible, and justify your answer.
   i) $x^{2^n} + 1 \in \mathbb{Q}[x]$.
   ii) $x_n^n + x_{n-1}^{n-1} + \cdots + x_2^2 + x_1 \in F[x_1, \ldots, x_n]$ for F any field.
   iii) $x^4 + 1 \in \mathbb{F}_p[x]$, p an odd prime (note that $p^2 \equiv 1 \pmod 8$).
   iv) $x^p + x^{p-1} + \cdots + x + 1 \in \mathbb{F}_p[x]$, p an odd prime.

5. Let R be a commutative ring with 1, and let $r_1, \ldots, r_n \in R$ satisfy $R = Rr_1 + \cdots + Rr_n$. If $M = \{(a_1, \ldots, a_n) \in R^n \mid a_1 r_1 + \cdots + a_n r_n = 0\}$, show that M is a projective R module.

6. Let K be a finite Galois extension of Q with $\mathrm{Gal}(K/\mathbb{Q}) \cong A_4$. How many subfields does K contain, what are their dimensions over Q, and which are Galois over Q?

① $|G| < \infty$, P a Sylow p-sg., $N_G(P)$ normalizer

(i) Show $N_G(P) = N_G(N_G(P))$:

See that $P \trianglelefteq N_G(P) \trianglelefteq N_G(N_G(P))$. Choose $g \in N_G(N_G(P))$. Since $P \leq N_G(P)$ is a Sylow p-subgroup of $N_G(P)$ as well and $N_G(P)$ is closed under conjugation with g since $N_G(P) \trianglelefteq N_G(N_G(P))$, gPg^{-1} is also a Sylow p-sg. of $N_G(P)$; but $P \trianglelefteq N_G(P)$ means there is only one Sylow p-sg. here, hence $P = gPg^{-1}$, hence $P \trianglelefteq N_G(N_G(P))$

Now, $N_G(P)$ is the largest subgrp of G s.t. P is normal in it, hence $\underline{N_G(P) = N_G(N_G(P))}$

(ii) If $K \trianglelefteq G$ and $P \leq K$, show: $KN_G(P) = G$

First, since $P \leq K$, P is also a Sylow p-sg. of K. Now for $g \in G$, gPg^{-1} is also a Sylow p-sg. of K since K is closed under conj. All Sylow p-sg. are conjugate in K, hence $\exists\, k \in K$ such that $gPg^{-1} = kPk^{-1}$; hence for $s \in P$, $\exists\, t \in P$ such that

$$gsg^{-1} = ktk^{-1} \Rightarrow g = ktk^{-1}gs^{-1}$$

$k \in K$, $t \in P \leq K$

$\Rightarrow kt \in K$

$$k^{-1}gs^{-1}Psg^{-1}k$$
$$= k^{-1}gPg^{-1}k \quad \text{since } s \in P$$
$$= k^{-1}kPk^{-1}k$$
$$= P \Rightarrow k^{-1}gs^{-1} \in N_G(P)$$

(iii) If no proper sg. of G is its own normalizer, show that $Z(G) \neq 1$:

Recall $N_G(P) = N_G(N_G(P))$, hence $N_G(P)$ cannot be proper, hence $N_G(P) = G$, hence $P \trianglelefteq G$ $\forall$ Sylow p-sgs P.

Hence G is a direct sum of its Sylow p-subgroups, all of which have nontrivial center since p-groups.

(2) Up to isomorphism describe all finitely-generated abelian groups which satisfy all of these:

(1) $G \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^2$

(2) $G \otimes_{\mathbb{Z}} (\mathbb{Z}_7) \cong (\mathbb{Z}_7)^3$

(3) $G \otimes_{\mathbb{Z}} (\mathbb{Z}_p) \cong (\mathbb{Z}_p)^2$ for $p \neq 7$.

$G \cong \mathbb{Z}^m \oplus G_T$

(1): $(\mathbb{Z}^k \oplus G_T) \otimes \mathbb{Q} \cong (\mathbb{Z}^k \otimes \mathbb{Q}) \oplus (G_T \otimes \mathbb{Q})$ ($G_T \otimes \mathbb{Q} \to 0$)

$\cong \mathbb{Q}^m \cong \mathbb{Q}^2 \Rightarrow \underline{m = 2}$

$G_T \cong \mathbb{Z}_{p_1^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{e_k}}$

(2): $(\mathbb{Z}^2 \oplus G_T) \otimes_{\mathbb{Z}} \mathbb{Z}_7 \cong (\mathbb{Z}^2 \otimes \mathbb{Z}_7) \oplus (G_T \otimes_{\mathbb{Z}} \mathbb{Z}_7)$

$\cong \mathbb{Z}_7^2 \oplus (G_T \otimes_{\mathbb{Z}} \mathbb{Z}_7)$.

Here $G_T \otimes_{\mathbb{Z}} \mathbb{Z}_7 \cong \mathbb{Z}_7$.

$\Rightarrow (\mathbb{Z}_{p_1^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{e_k}}) \otimes \mathbb{Z}_7 \cong (\mathbb{Z}_{p_1^{e_1}} \otimes \mathbb{Z}_7) \oplus \cdots \oplus (\mathbb{Z}_{p_k^{e_k}} \otimes \mathbb{Z}_7)$

$\cong \mathbb{Z}_7$

$\Rightarrow p_i \neq 7$ for only one $i = 1, \ldots, k$.

Re-index $G_T \cong \mathbb{Z}_{7^{e_1}} \oplus \mathbb{Z}_{p_2^{e_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{e_k}}$, $\gcd(p_i, 7) = 1$.

(3): $G \otimes_{\mathbb{Z}} (\mathbb{Z}_p) \cong (\mathbb{Z}_p)^2$ for $p \neq 7$.

$(\mathbb{Z}^2 \oplus G_T) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong (\mathbb{Z}^2 \otimes \mathbb{Z}_p) \oplus (G_T \otimes \mathbb{Z}_p)$

$\cong \mathbb{Z}_p^2 \oplus (G_T \otimes \mathbb{Z}_p)$.

$\Rightarrow G_T \otimes \mathbb{Z}_p \cong 0$, hence all summands orders have prime to p

For all $p \neq 7$, hence only a 7-primary summand:

$\Rightarrow \boxed{G \cong \mathbb{Z}^2 \oplus \mathbb{Z}_{7^e}}$

⑤ R commutative ring with 1 and let $r_1, \dots, r_n \in R$ satisfy $R = Rr_1 + \cdots + Rr_n$. If $M = \{(a_1, \dots, a_n) \in R^n : a_1 r_1 + \cdots + a_n r_n = 0\}$, show M is projective R-module.

Consider the map $\varphi : R^n \longrightarrow R$

$$(a_1, \dots, a_n) \mapsto a_1 r_1 + \cdots + a_n r_n.$$

Now, since $R = Rr_1 + \cdots + Rr_n$, $\exists (s_1, \dots, s_n) \in R^n$ such that $s_1 r_1 + \cdots + s_n r_n = 1$, hence $\varphi(s_1, \dots, s_n) = 1$, hence φ surjective.

Now note that $\ker \varphi = M$ by construction; ~~then we get $R^n / M \cong R$ by first isom.~~

Now consider $0 \to M \hookrightarrow R^n \xrightarrow{\varphi} R \to 0$

since $\ker \varphi = M$ this sequence is exact, and R is a free R-module, hence it splits and we get

$$R^n \cong M \oplus R,$$ hence M projective.

⑥ $K/\mathbb{Q}$ finite galois ext, $Gal(K/\mathbb{Q}) \cong A_4$.
How many subfields does K contain, what are their dimensions over $\mathbb{Q}$, and which are Galois?

Fundamental theorem of Galois theory tells us that the subgps of A_4 correspond to the subfields of K.
$|A_4| = 12 = 3 \cdot 2^2$.

There is no order 6 subgp, but we have subgps of order 3, 2 and 2^2.

- The order 2^2 sg. is normal (unique one normal Sylow 2-sg.) and $\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This corresponds to the Galois subextension. This is only normal sg., hence only Galois subext. order 4 $\Rightarrow$ index 3 $\Rightarrow [K^{\mathbb{Z}_2 \oplus \mathbb{Z}_2} : \mathbb{Q}] = 3$.
- There are ③ order 2 subgps given by the double transpositions; thereby index 6, hence $[K^{\mathbb{Z}_2} : \mathbb{Q}] = 6$.
- There are ④ order 3 subgps given by 3-cycles; index 4, hence $[K^{\mathbb{Z}_3} : \mathbb{Q}] = 3$. In each

Total of 8 subextensions