ALGEBRA QUALIFYING EXAM NOVEMBER, 1999

Partial credit is given for partial solutions.

1. For p and q distinct primes show that any group of order p^2q is solvable.

2. Let G be a finite Abelian group so that whenever H and K are subgroups of G of the same order then $H \cong K$ as groups. Describe the possible structures of G. If $|G| = 2^3 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$, up to isomorphism how many possibilities are there for G?

3. Let $p_1, \ldots, p_k$ be distinct primes in $\mathbf{Z}$ and set $F = \mathbf{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_k}) \subseteq \mathbf{R}$.
   i) Show that F is a Galois extension of $\mathbf{Q}$ with $\mathrm{Gal}(F/\mathbf{Q}) \cong (\mathbf{Z}/2\mathbf{Z})^k$.
   ii) Show that $F = \mathbf{Q}(\sqrt{p_1} + \cdots + \sqrt{p_k})$.

4. For F a field and $R = F[x_1, \ldots, x_n]$ let M be a finitely generated R module. Show that there are positive integers s and t and an exact sequence of R modules $0 \to K \to R^s \to R^t \to M \to 0$.

5. If I is a nonzero ideal of $R = \mathbf{C}[x_1, \ldots, x_n]$ which is not maximal then if R/I is a domain, show that $\dim_C R/I$ must be infinite.

6, If $R \neq \{0\}$ is a finite ring so that each $r \in R$ satisfies the polynomial $x^8 = x$, describe the possible structures of R.

ALGEBRA QUALIFYING EXAM   MAY, 1999

Partial credit is given for partial solutions.

1. Let G be a group of order $1705 = 5\cdot 11\cdot 31$. Describe the possible structures of G up to isomorphism.

2. Let G be a group and $N \lhd G$. Show: i) G is solvable $\Leftrightarrow$ N and G/N are solvable; and ii) if $|G| = p^n$ for p a prime then G is solvable.

3. Let $f(x) \in \mathbf{Q}[x]$ be irreducible with deg $f = p$, an odd prime, and let $K \subseteq \mathbf{C}$ be a splitting field for f(x) over $\mathbf{Q}$. Suppose that f(x) has exactly two roots in $\mathbf{C} - \mathbf{R}$. Prove that $\mathrm{Gal}(K/\mathbf{Q}) \cong S_p$.

4. Using methods of algebraic geometry show that there is a fixed $m > 0$ so that for any linear polynomial $f(x,y,z,t) = ax + by + cz + dt$, $f(x,y,z,t)^m \in (x^{19}y^{23}z^{29}t^{31}, x^3 + y^5, y^7 + z^{11}, z^{13} + t^{17}) \subseteq \mathbf{C}[x,y,z,t]$.

5. Let $A = \mathbf{C}[x, \sigma]$ be the twisted polynomial ring over $\mathbf{C}$ where σ is complex conjugation. The elements of A are the polynomials $p(x) = c_nx^n + \cdots + c_1x + c_0$ which add in the usual way but with multiplication given by $xa = \sigma(a)x = \bar{a}x$, and extended by the associative and distributive laws. The general expression for products is $\sum a_ix^i \sum b_jx^j = \sum(\sum_{i+j=k} a_i\sigma^i(b_j))x^k$.

   i) Find the center of A.
   ii) Is the center of A a Noetherian ring (and why)?
   iii) Show that A is a left and a right Noetherian ring.

6. Let R be a right Artinian ring so that each $r \in R$ satisfies $r^5 = r$.

   i) Show that R is a finite ring.
   ii) Show that there is some $m \geq 1$ so that R has exactly 2^m elements satisfying $x^2 = x$.
   iii) Using m in ii), find the possible values for $|R| = \mathrm{card}(R)$.

Written Qualifying Exam, Algebra, Nov. 1998

Directions. Partial credit in units of 1/4 is given for partial solutions.

1. Let G be a group of order $p^a q^b$, p, q distinct primes and a, b positive integers. Prove that if $q < p$ and the order of q mod p exceeds b then G is solvable.

2. Let G be a finitely generated abelian group (i.e., a finitely generated $\mathbb{Z}$-module).

   a). Prove that G has no elements of order p, p a prime, if and only if $G \otimes_{\mathbb{Z}} \mathbb{Z}_P \cong \mathbb{Z}_P^r$ for some positive integer r, $\mathbb{Z}_P$ is the local ring of rational numbers with denominator prime to p.

   b). Prove that G is projective if and only if there is an integer r such that $G \otimes_{\mathbb{Z}} H \cong H^r$ for all abelian groups H.

3. Let $\mathbb{F}_{p^n}$ be a finite field with p^n elements, p a prime. Recall that the norm map $N : \mathbb{F}_{p^n}$ is defined by $N(x) = \prod_{g \in Gal_{\mathbb{F}_p}\mathbb{F}_{p^n}} g(x)$ and the trace map is defined by $T(x) = \sum_{g \in Gal_{\mathbb{F}_p}\mathbb{F}_{p^n}} g(x)$. Determine the image of each of these maps, show that the kernel of the norm map is $\{x/g(x) : x \in \mathbb{F}_{p^n}^{\times}, g \in Gal_{\mathbb{F}_p}\mathbb{F}_{p^n}\}$ and that the kernel of the trace map is $\{x - g(x) : x \in \mathbb{F}_{p^n}, g \in Gal_{\mathbb{F}_p}\mathbb{F}_{p^n}\}$.

4. Let R be a subring of $\mathbb{C}[x_1, \ldots, x_n]$ containing $\mathbb{C}$ and assume that the field of quotients of R is $\mathbb{C}(x_1, \ldots, x_n)$. Show that there are polynomials $f_1, \ldots, f_x \in \mathbb{C}[x_1, \ldots, x_n]$ such that $d\mathbb{C}[x_1, \ldots, x_n] \subset R$ if and only if $d \in I = f_1\mathbb{C}[x_1, \ldots, x_n] + \cdots + f_s\mathbb{C}[x_1, \ldots, x_n]$. In addition, show that I cannot be a maximal ideal in $\mathbb{C}[x_1, \ldots, x_n]$.

5. Maschke's theorem implies that the group algebra $k[G]$ over a field k of characteristic zero is semisimple when G is a finite group. Using this fact,

   a). Determine the structure of $\mathbb{C}[S_3]$, S_3 the symmetric group on three symbols.

   b). An epimorphism of groups, $\phi : G \to H$, induces an epimorphism $\Phi : k[G] \to k[H]$ on the corresponding group rings over k. Prove that if k has characteristic 0 and G is finite then $k[H]$ is a ring direct summand of $k[G]$.

6. Determine the galois group of $x^4 - p$ over the rationals, p a prime, and determine all subfields of its splitting field which are normal over the rational numbers.

ALGEBRA QUALIFYING EXAM   MAY, 1998

Partial credit is given for partial solutions.

1. Let G be a group of order 105. Show that G contains a normal subgroup of index 3 and determine how many possibilities there are for the structure of G, up to isomorphism. Show that G has a nontrivial center.

2. For a prime integer p, a group G is called p-divisible if the function $f_p: G \to G$ given by $f_p(g) = g^p$ is surjective (i.e. onto). If G is Abelian and p-divisible, show that G is finitely generated if and only if G is finite with order relatively prime to p.

3. Let $\mathbf{Q} \subseteq M \subseteq \mathbf{C}$ with M a finite dimensional Galois extension of $\mathbf{Q}$, the rational numbers. If for all subfields $\mathbf{Q} \subseteq L \subseteq M$, $[L:\mathbf{Q}]$ is even, what can the order of $\mathrm{Gal}(M/\mathbf{Q})$ be? In this case, show that M embeds in a radical extension of $\mathbf{Q}$.

4. Let $\mathbf{C}[x_1, x_2, \ldots, x_n] = R$ and let $f(X) = f(x_1, x_2, \ldots, x_n) \in R$ be irreducible. Given $g(X), h(X) \in R$ so that $g(\alpha) - h(\alpha) = 0$ for all $\alpha \in \mathbf{C}^n$ satisfying $f(\alpha) = 0$, show that $g(X) + (f(X)) = h(X) + (f(X))$ in $R/(f(X))$.

5. Let R be a commutative Noetherian ring with 1. Prove that R is isomorphic to a finite direct sum of fields if and only if every (ring) homomorphic image of R is projective as an R module.

6. Let F be a finite field and let A be an F subalgebra of $M_n(F)$.
   a) If A is a domain, show that $\dim_F A \leq n$.
   b) If A is simple with $F \cdot I_n$ as its center, show that $\sqrt{\dim_F A}$ is an integer and divides n.

Algebra Qualifying, Prequalifying, and M.A. Comprehensive Exam, 11/97

Directions: Work any 5. Partial credit in units of 1/4 is given for partial solutions.

1. Let G be a finite group, N a normal subgroup, and P a p-Sylow subgroup of N.

a). Show that $G = NN_G(P)$, $N_G(P) = \{g \in G : gPg^{-1} = P\}$ the normalizer of P in G.

b). Let $\Phi(G)$ denote the intersection of the proper maximal subgroups of G. Show that $\Phi(G)$ is normal in G and if H is a subgroup such that $G = \Phi(G)H$ then $H = G$.

c). Show that $\Phi(G)N_G(P) = G$ and that every p-Sylow subgroup P of $\Phi(G)$ is normal in G.

2. View the $n \times n$ matrix T over the ring of integers $\mathbb{Z}$ as a linear transformation on $\mathbb{Z}^n$; that is, $T(X) = XT$, the matrix product of $X = (x_1, ..., x_n)$ and T. Set $Im(T) = \{T(X) : X \in \mathbb{Z}^n\}, Ker(T) = \{X \in \mathbb{Z}^n : T(X) = 0\}$.

a). Show that $\mathbb{Z}^n = Ker(T) \oplus Im(T)$.

b). What is the structure of the abelian group $\mathbb{Z}^n/Im(T)$ if $T^2 = pI_n, p$ a prime and I_n the $n \times n$ identity matrix.

3. Let $\mathbb{Q} \subset F \subset \mathbb{C}$ where F is the field generated over the rationals $\mathbb{Q}$ by all roots of unity in the field $\mathbb{C}$ of complex numbers. Let $a_1, \ldots, a_k \in \mathbb{Q}, p_1 < \cdots < p_k$ primes, and set $M = F(a_1^{1/p_1}, \ldots, a_k^{1/p_k})$.

a). Show that M is a Galois extension of F.

b). Describe the Galois group of M over F.

c). For any subfield $F \subset K \subset M$, show that $K = F(S)$ for some subset S of $\{a_1^{1/p_1}, \ldots, a_k^{1/p_k}\}$.

4. let $\overline{\mathbb{F}}_p$ denote the algebraic closure of the finite field $\mathbb{F}_p$ with p elements, p a prime, and let $\mathbb{F}_{p^n}$ denote the subfield of $\overline{\mathbb{F}}_p$ with p^n elements.

a). For $x \in \overline{\mathbb{F}}_p$, show that $x^{\frac{p^n-1}{p-1}} \in \mathbb{F}_p$ if and only if $x \in \mathbb{F}_{p^n}$.

b). Let $F(n) = \{x \in \overline{\mathbb{F}}_p : x^n \in \mathbb{F}_p\}$. Show that $F(n)$ is finite and $F(pn) = F(n)$.

5. Let F denote a field, σ, τ automorphisms of F generating an abelian subgroup H of $\mathrm{Aut}(F)$ of finite order s. The twisted polynomial ring $F_H[x,y]$ consists of all polynomial expressions $\sum_{i,j=0}^{m} f_{ij}x^iy^j, f_{ij} \in F$ in commuting indeterminates x, y over F subject to the usual polynomial addition and multiplication except that $xf = \sigma(f)x, yf = \tau(f)y$. Let Z denote the center of $F_H[x,y]$.

a). Show that Z contains $F_0[x^s, y^s]$, F_0 the subfield of F fixed by H.

b). Show that $F_H[x,y]$ is Noetherian and Z is Noetherian.

6. Let I denote an ideal in $\mathbb{C}[x_1, \ldots, x_n], \mathbb{C}$ the field of complex numbers and suppose that I is the intersection of k maximal ideals. Show that if $k < n$ then I contains a homogeneous linear polynomial $a_1x_1 + \cdots + a_nx_n$ with $a_i \neq 0$ for some $i, 1 \leq i \leq n$. Give an example to show that this can fail for $k \geq n$.

7. Let R is a finite dimensional, semisimple $\mathbb{C}$-algebra, $\mathbb{C}$ the field of complex numbers, and for $r \in R - \{0\}$, let $m_r(x) \in \mathbb{C}[x]$ denote the monic polynomial of least degree such that $m_r(r) = 0$; i.e., the minimal polynomial for r. Show that R is commutative if and only if $m_r(x)$ has no multiple roots $\forall\, r \in R - \{0\}$, and R is noncommutative if and only if $\deg m_r(x) < \dim R/k\, \forall\, r \in R - \{0\}$.

ALGEBRA QUALIFYING EXAM MAY, 1997

Partial credit is given for partial solutions.

1. Up to isomorphism, describe all group of order 495.

2. Let $x^4 - 7 \in F[x]$ for $F \subseteq \mathbf{C}$. If $F \subseteq M \subseteq \mathbf{C}$ and M is a splitting field for $x^4 - 7$ over F, find Gal(M/F): when $F = \mathbf{Q}$; when $F = \mathbf{Q}[\sqrt{7}]$; and when $F = \mathbf{Q}[i]$, with $i^2 = -1$.

3. Let M be a finitely generated F[x] module (F a field). If every submodule of M has a complement, describe the structure of M in terms of F[x]. (Recall that a submodule H of a module M has a complement if there is a submodule H' so that $M \cong H \oplus H'$; i.e. $H + H' = M$ and $H \cap H' = (0)$.)

4. Show that some power of $(x + y)(x^2 + y^4 - 2)$ is in the ideal of $\mathbf{C}[x,y]$ generated by $x^3 + y^2$ and $y^3 + yx$.

5. Let R be a commutative Noetherian ring with no nonzero nilpotent element. Set A = {ann I | I is a nonzero ideal of R} and M = {maximal elements in A}. Prove that R embeds in a direct sum of finitely many domains as follows:
   a) Show that the elements of M are prime ideals in R.
   b) For $P \neq Q$ in M, show ann $Q \subseteq P$.
   c) Show that M is finite (consider sums of ann P_i for $P_i \in M$).
   d) Show that the intersection of the elements in M is zero.

6. Let R be a finite dimensional algebra over the field F. Assume that for every $r \in R$ there some $g(x) \in F[x]$, depending on *r*, so that $r + g(r)r^2 = 0$. Determine the structure of R.

ALGEBRA QUALIFYING EXAM (MATH 510AB)

FALL 1996

(1) Let G be a group with $|G| = 585$. Show that G contains a normal cyclic subgroup of prime index. Describe G up to isomorphism. Show that $Z(G) \neq e$ and has composite order.

(2) For any $n > 3$, show that each element of the symmetric group S_n is a product of permutations, each having no fixed point in $\{1, 2, \ldots, n\}$.

(3) Show that any cyclic group G with square free order ($a > 0$ and $a^2 \Big| |G|$ implies that $a = 1$) is the Galois group over $\mathbb{Q}$ of some field extension $K \supseteq \mathbb{Q}$. (Hint: For a suitable $n > 0$, consider $x^n - 1 \in \mathbb{Q}[x]$.)

(4) Let $(\mathbb{Q}, +)$ be the rational numbers under addition.
   (a) Show that $(\mathbb{Q}, +)$ is not a finitely generated Abelian group.
   (b) Show that any finitely generated $\mathbb{Z}$ submodule of $\mathbb{Q}$ is free.
   (c) Determine if $(\mathbb{Q}, +)$ is a free $\mathbb{Z}$ module.
   (d) Is $(\mathbb{Q}, +)$ a projective $\mathbb{Z}$ module.

(5) Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and let I and J be ideals of R satisfying: for all $\alpha \in \mathbb{C}^n$, $f(\alpha) = 0$ for all $f \in I$ in and only if $g(\alpha) = 0$ for all $g \in J$.
   (a) Show that $(I + J)/I$ is a nil ring.
   (b) Show that $(I + j)/I$ is a nilpotent ring. (Note that $(I + J)/I$ is an ideal of R/I.)

(6) If R is a finite ring and $x^5 = x$ for all $x \in R$, describe the structure of R.

ALGEBRA QUALIFYING EXAM MAY, 1996

Partial credit is given for partial solutions.

1. Let G be a finite group and p a prime number. Let P be a p-Sylow subgroup of G and denote the normalizer of P in G by $N_G(P)$.
   i) Show that $N_G(P) = N_G(N_G(P))$.
   ii) If K is a normal subgroup of G and K contains P, show that $G = KN_G(P)$.
   iii) If no proper subgroup of G is its own normalizer, show that the center of G is not trivial.

2. Up to isomorphism describe all finitely generated Abelian groups which satisfy all of the following properties: i) $G \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}^2$; ii) $G \otimes_{\mathbf{Z}} (\mathbf{Z}/7\mathbf{Z}) \cong (\mathbf{Z}/7\mathbf{Z})^3$; and iii) for any prime $p \neq 7$, $G \otimes_{\mathbf{Z}} (\mathbf{Z}/p\mathbf{Z}) \cong (\mathbf{Z}/p\mathbf{Z})^2$.

3. Let R be a left Artinian ring with Jacobson radical J(R). If $R \neq J(R)$ show that R is a left Noetherian ring.

4. Determine if each of the following polynomials is irreducible, and justify your answer.
   i) $x^{2^n} + 1 \in \mathbf{Q}[x]$.
   ii) $x_n^n + x_{n-1}^{n-1} + \cdots + x_2^2 + x_1 \in F[x_1, \ldots, x_n]$ for F any field.
   iii) $x^4 + 1 \in F_p[x]$, p an odd prime (note that $p^2 \equiv 1 \pmod 8$).
   iv) $x^p + x^{p-1} + \cdots + x + 1 \in F_p[x]$, p an odd prime.

5. Let R be a commutative ring with 1, and let $r_1, \ldots, r_n \in R$ satisfy $R = Rr_1 + \cdots + Rr_n$. If $M = \{(a_1, \ldots, a_n) \in R^n \mid a_1r_1 + \cdots + a_nr_n = 0\}$, show that M is a projective R module.

6. Let K be a finite Galois extension of $\mathbf{Q}$ with $\mathrm{Gal}(K/\mathbf{Q}) \cong A_4$. How many subfields does K contain, what are their dimensions over $\mathbf{Q}$, and which are Galois over $\mathbf{Q}$?

ALGEBRA QUALIFYING EXAM **NOVEMBER, 1995**

Partial credit is given for partial solutions.

1. Up to isomorphism, determine all groups of order $7^2 \cdot 11^2 \cdot 19$.

2. Let G be a finite Abelian group and recall that the *exponent* of G is the smallest positive integer n so that $g^n = e_G$ for all $g \in G$. Show that G is a cyclic group if and only if the order and exponent of G are equal.

3. An ideal in a commutative ring is called *irreducible* if it cannot be written as an intersection of finitely many properly larger ideals. If $A = \mathbf{Z}[x_1, ..., x_n]$ is the polynomial ring in n variables over $\mathbf{Z}$, show that any ideal of A is an intersection of finitely many irreducible ideals.

4. Let K be a splitting field over $\mathbf{Q}$ of the polynomial $x^{11} - 17$. Show that $Gal(K/\mathbf{Q})$ is isomorphic to the group of matrices $G = \{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \in GL(2,\mathbf{Z}/11\mathbf{Z}) \}$.

5. Let F be a field and M an irreducible (i.e. simple and nontrivial) $F[x_1, ..., x_n]$ module.
   i) If F is algebraically closed, show that $\dim_F M = 1$.
   ii) For any F show that $\dim_F M$ is finite.

6. Let R be a finite ring in which every element is a sum of nilpotent elements. Show that R is nilpotent. (Hint: what is the trace of a nilpotent element in $M_n(F)$ for F a field?)

ALGEBRA QUALIFYING EXAM (MATH 510AB)

SPRING 1995

(1) Let G be a finite group.
   (a) If $|G| = 2^s \cdot 7$ for $0 \leq s \leq 3$, then G is solvable.
   (b) Suppose $|G| = 112$ and G is not simple. Show that G is solvable.
   (c) If $|G| = 112$, show that G is not simple. *Hint*: If G were simple, show that G embeds into S_7, and then show that G embeds, in fact, into A_7.

(2) Suppose that a group G is the direct sum of k cyclic groups, each of prime power order. If H is a subgroup of G containing nontrivial subgroups $H_1, \ldots, H_s$, whose sum is direct, show that $s \leq k$.

(3) Let $f(x) = (x^3 - 3)(x^2 + 1) \in \mathbb{Q}[x]$. Denote by K the splitting field of $f(x)$ over $\mathbb{Q}$, and by G the Galois group of K over $\mathbb{Q}$.
   (a) Find $|G|$.
   (b) Show that G has a normal subgroup of order 2.
   (c) Show that the 3-Sylow subgroup of G is normal. *Hint*: K contains all 12-th roots of unity.
   (d) Show that G has a central element of order 2.

(4) Let R be a right Noetherian ring (with 1). Prove that R has a unique maximal nilpotent ideal $P(R)$. Show that the polynomial ring $R[x]$ must also have a unique maximal nilpotent ideal $P(R[x])$, and that $P(R[x]) = P(R)[x]$.

(5) Let M be a maximal ideal in the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$. Show that there are only finitely many maximal ideals in $\mathbb{C}[x_1, \ldots, x_n]$ containing M. *Hint:* Show first that for each i there is a polynomial $f_i(y) \in \mathbb{Q}[y]$ such that $f_i(x_i) \in M$.

(6) Let G be a finite group, and $\mathbb{C}[G]$ it group algebra. Define a bijection $^* : \mathbb{C}[G] \to \mathbb{C}[G]$ as follows: For $x = \sum_{g \in G} a_g g \in \mathbb{C}[G]$, set $x^* = \sum_{g \in G} \overline{a}_g g^{-1}$. One calls x summetric if $x = x^*$.
   (a) Given $x, y \in \mathbb{C}[G]$, show that $(x^*)^* = x$, and $(xy)^* = y^* x^*$.
   (b) Given $x \in \mathbb{C}[G]$, show that xx^* is symmetric, and that $xx^* = 0$ if and only if $x = 0$.
   (c) Show that nonzero symmetric elements are not nilpotent.
   (d) Assume that $\mathbb{C}[G]$ has no nonzero nilpotent elements. Show that G is abelian.

(7) Let K be a field extension of $k = \mathbb{F}_{p^r}$ of degree n. Let σ be the automorphism of K given by $\sigma(a) = a^{p^r}$ for $a \in K$.
   (a) Let x be an element of K such that both $x + \sigma(x)$ and $x\sigma(x)$ belong to k. Show that $[k(x) : k] \leq 2$. Moreover, show that $[k(x) : k] = 2$ if and only if $\sigma(x) \neq x$.
   (b) Set
   $$F = \{x \in K | x + \sigma(x), x\sigma(x) \in k\}$$
   Show that F is a subfield of K, and that $[F : k] \leq 2$. Moreover, show that $[F : k] = 2$ if and only if $2|n$.

ALGEBRA QUALIFYING EXAM (MATH 510AB)

SPRING 1993

(1) Find up to isomorphism all groups of order $3 \cdot 7 \cdot 19 \cdot 37$.

(2) Let D be a commutative domain with multiplicative identity 1 and assume that the additive group D is finitely generated. Prove
   (a) Characteristic $K = 0$ if and only if $(D, +)$ is a free abelian group.
   (b) If $\exists$ an integer $n > 1$ such that $f : D \to D$ defined by $x \mapsto nx$ is onto then D is a finite field.
   (c) If M is a maximal ideal in D then $M \cap i(\mathbb{Z}) = \pi(\mathbb{Z})$ for some prime p, $i(\mathbb{Z}) = \{n \cdot 1, n \in \mathbb{Z}\}$.

(3) Let $f(x) \in \mathbb{Q}(x)$ be irreducible with $\deg f = n$. Let $M \subset \mathbb{C}$ be a splitting field for $f(x)$ over $\mathbb{Q}$.
   (a) Show that if $\mathrm{Gal}(M/\mathbb{Q})$ is abelian then every subfield field of M is Galois over $\mathbb{Q}$.
   (b) Show that if $\mathrm{Gal}(M/\mathbb{Q})$ is abelian then $[M : \mathbb{Q}] = n$.

   Let $f(x) \in \mathbb{F}_{p^n}[x]$ be irreducible of degree t, $\mathbb{F}_{p^n}$ a field with p^n elements, p a prime.
   (a) Show that $\mathbb{F}_{p^{nt}}$ is a splitting field of f over $\mathbb{F}_{p^n}$.
   (b) For $n = 1$, show that $f(x)$ divides $x^{p^m} - x$ if and only if t divides m.
   (c) How many distinct irreducibles in $\mathbb{F}_2[x]$ have degree 5?

(4) Let $f_i(x, y) = a_i x^2 + b_i xy + c_i y^2 \in \mathbb{C}[x, y]$, $1 \leq i \leq n$. Show that there exists $(u, v) \in \mathbb{C}^2$ such that $u^2 + v^2 = 1$, but $f_i(u, v) \neq 0 \ \forall \ i = 1, \ldots, n$.

(5) Given the linear equation $a_1 X_1 + \ldots + a_t X_t = 0$, $a_i \in A = k[x_1, \ldots, x_m]$ and k a field, prove that there are solutions $Y_1, \ldots, Y_q \in A^t$ such that for each solution Y, there exists $b_1, \ldots, b_q \in A$ such that $Y = \sum_{i=1}^{q} b_i Y_i$. If $A = \mathbb{Z}$, prove that you can take $q = t - 1$.

(6) Let x denote a fixed non zero vector in $\mathbb{C}^3$ and A_x denote the ring of matrices $T \in \mathbb{M}_3(\mathbb{C})$ such that $xT = 0$.
   (a) Prove that $A_{xU} \cong A_x$ for any $U \in \mathrm{GL}_3(\mathbb{C})$, hence $A_x \cong A_y$ for any non zero $y \in \mathbb{C}^3$.
   (b) Prove that $\{(a_{ij}) \in A_{(1,0,0)} : a_{ij} = 0 \text{ for } j > 1\}$ is nilpotent ideal in $A_{(1,0,0)}$.
   (c) Prove that the Jacobson radical $J(A_x)$ is not zero and that $A_x / J(A_x) \cong \mathbb{M}_c(\mathbb{C})$.

ALGEBRA QUALIFYING EXAM (MATH 510AB)

FALL 1992

(1) If G is a group of order $2^3 \cdot 19 \cdot 23$ show that G has a normal subgroup of order $4 \cdot 19 \cdot 23$ and the center of G contains an element of order 2.

(2) Let $T : \mathbb{Z}^n \to \mathbb{Z}^m$ be a group homomorphism.
   (a) If T is onto, show that $\mathbb{Z}^n \cong \operatorname{Ker} T \oplus \mathbb{Z}^m$.
   (b) Prove that T is injective if and only if $m \geq n$ and $\dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/T(\mathbb{Z}^n)) = m - n$.

(3) Let F be a finite Galois extension of the field k, A subfield $k \subset L \subset F$ is abelian if L is a Galois extension of k and $\mathrm{Gal}(L/k)$ is abelian.
   (a) Prove there is a unique maximal abelian subfield of F.
   (b) Prove that if L, J are abelian extensions of k then their composite is an abelian extension of k.

(4) Let L be the splitting field of $x^{98} - 1$ over $\mathbb{Q}$. Find $[L : \mathbb{Q}]$, $|\,\mathrm{Gal}(L/\mathbb{Q})|$, the structure of $\mathrm{Gal}(L/\mathbb{Q})$, the number of subfields of L and the subfields which are normal over $\mathbb{Q}$.

(5) (a) Let $M_2 \xrightarrow{g} M_3 \xrightarrow{h} M_4 \to 0$ be an exact sequence of A-modules, A a ring and $f : M_1 \to M_2$ an A-module homomorphism. Prove that $M_1 \xrightarrow{g \circ f} M_3 \xrightarrow{h} M_4 \to 0$ is exact if and only if $f(M_1) + \ker g = M_2$.
   (b) Show that whenever $A = k[x, y]$, k a field, and I is an ideal in A then there is an exact sequence $A^m \to A^n \to I \to 0$ for some positive integers m, n.

(6) For an ideal I in $A = \mathbb{C}[x, y, z]$ set $Z_{xy}(I) = \{(a, b) \in \mathbb{C}^2 : f(a, b, z) = 0 \text{ for all } f \in I\}$
   (a) Prove that I maximal implies Z_{xy} is empty.
   (b) Prove that $Z_{xy}(I) \times \mathbb{C} = Z(I) = \{(a, b, c) \in \mathbb{C}^3 : f(a, b, c) = 0 \forall f \in I\}$ if and only if $\operatorname{rad} I = JA$, where $J = \{f(x, y) : f(a, b) = 0 \forall (a, b) \in Z_{xy}\}$.

(7) Describe up to isomorphism all semi-simple $\mathbb{C}$0subalgebras of $M_4(\mathbb{C})$, the ring of 4×4 matrices over $\mathbb{C}$. (Note that if A, B are $\mathbb{C}$-algebras and $\alpha \in A$, $\beta \in B$ have minimal polynomials, f, g respectively then $(\alpha, \beta) \in A \oplus B$ has minimal polynomial $h = \operatorname{lcm}(f, g)$.

ALGEBRA QUALIFYING EXAM (MATH 510AB)

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(1) Let G be a group with $|G| = 5 \cdot 7^2 \cdot 17$. Determine the possible structures for G.

(2) Let G be a finitely generated abelian group $G \neq \{1\}$, n a positive integer and let $\varphi_n : G \to G$ be the homomorphism defined by $\varphi_n(g) = g^n$. Prove
  (a) G is not divisible, i.e. $\exists n$ such that φ_n is onto.
  (b) G is finite if and only if $\exists n$ such that φ_n is the trivial map, i.e. $\varphi_n(g) = 1$ for all $g \in G$.
  (c) G is free abelian group if and only if φ_n is 1-1 for all positive integers n.
  (d) G is finite if and only if φ_n is an isomorphism for some positive integer $n > 1$.

(3) Let $k = \mathbb{Q}(\zeta_{15})$, where ζ_{15} is a primitive 15-th root of unity. What is the galois group of $k/\mathbb{Q}$? How many subfields does k have? List all subfields (recall that the field $\mathbb{Q}(\zeta_p)$ of p-th roots of unity, p a prime, contains the subfield $\mathbb{Q}(\sqrt{p})$ if $p \equiv 1 \bmod 4$ and $\mathbb{Q}(\sqrt{-p})$ if $p \not\equiv 1 \bmod 4$).

(4) Let $F = \mathbb{F}_{p^n}$ be a field of p^n elements. For $1 \leq k \leq n$ set $L_k = \{a \in F : a^{p^k} = a\}$. Show that each L_k is a subfield of F, that $\{L - k : 1 \leq k \leq n\}$ is the set of all subfields of F, and for n greater than 2, $L_i = L_j$ for some $1 \leq i < j \leq n$.

(5) Let A be a commutative noetherian ring, M a noetherian A-module.
  (a) Prove that $M \otimes_A A[x]$ is a noetherian $A[x]$-module.
  (b) If A is a commutative Noetherian domain with 1, and $0 \neq y \in A$, a nonunit. Show that $y = a_1 a_2 \ldots a_k$ with each $a_i \in A$ irreducible.
  (c) Let $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over the complex numbers. An (irreducible) hypersurface in $\mathbb{C}^n$ is the solution set $Z(f)$ of $f(x) = 0$, f an irreducible polynomial in $\mathbb{C}[X]$. Let $\mathcal{F}(Z(f))$ denote the ring of complex valued polynomial functions on the hypersurface $Z(f)$; i.e. $h \in \mathcal{F}(Z(f))$ if and only if $\exists g \in \mathbb{C}[X]$ such that $h(T) = g(T)$ for all $T \in Z(f)$. Prove that $\mathcal{F}(Z(f)) \cong \mathbb{C}[X]/f(X)\mathbb{C}[X]$.
  (d) Let A be a finite dimensional semi-simple algebra over $\mathbb{C}$, and set $M_n(A)$ = ring of $n \times n$ matrices over A. Not that $M_n(M_m(A)) \cong M_{mn}(A)$ and $M_n(A \oplus B) \cong M_n(A) \oplus M_n(B)$.
  (a) Show that $M_2(A)$ is semi-simple.
  (b) If $\dim_{\mathbb{C}}(A)$ is prime, show that $M_2(A)$ is not simple.
  (c) If A is not commutative, there is a $t \in M_2(A)$ with $t^3 \neq 0$ and $t^4 = 0$.