Algebra qualifying exam, August 2025

Justify all arguments completely. Every ring R is assumed to have a unit $1_R \in R$. Given a field k, a k-algebra A is a ring which is equipped with a central ring homomorphism $k \to A$. Reference specific results whenever possible.

1. Classify up to isomorphism all groups of order $p \cdot q$, where $p < q$ are prime numbers.

2. Let k be a field. A k-algebra is called central if its center equals $k1_A \subset A$. Let A be a finite dimensional, central division k-algebra. Denote by $[A, A]$ the k-subspace of A spanned by the elements $ab - ba$ with $a, b \in A$. Prove $[A, A] \neq A$.

3. Let $\xi : R \to S$ be a surjective map between finitely generated commutative $\mathbb{C}$-algebras. Suppose that R is a domain, i.e. contains no zero divisors, and that for each maximal ideal $\mathfrak{m}$ in R there is a maximal ideal $\mathfrak{m}'$ in S for which $\xi^{-1}(\mathfrak{m}') = \mathfrak{m}$. Prove that ξ is an isomorphism.
[Hint: Take a surjection $\alpha : \mathbb{C}[x_1, \ldots, x_n] \to R$ and consider the kernels of the two maps α and $\xi\alpha$.]

4. Consider the polynomial $f(X) = X^{11} - 6 \in \mathbb{Q}[X]$. Prove that f is irreducible. Compute then the order of the Galois group G of f and determine whether G is solvable.

5. Let R be a commutative ring and let $\mathfrak{m}$ be a maximal ideal of R with the property that $1 + x$ is invertible for every $x \in \mathfrak{m}$. Show directly that $\mathfrak{m}$ is the unique maximal ideal of R.

6. Let $B : \mathbb{Z}^r \to \mathbb{Z}^r$ be an endomorphism of a free abelian groups. Considering B as a matrix, prove that B is invertible as a group homomorphism if and only if $\det(B) = \pm 1$.