ALGEBRA QUALIFYING EXAM, AUGUST 2023

Justify all arguments completely. Reference specific results whenever possible.

(1) Let $R = C_{an}(\mathbb{C})$ denote the ring of complex analytic functions on $\mathbb{C}$. We note that R is a domain. Prove one of the following statements:
  (a) The ring $R[t]$ of polynomials with coefficients in R is a PID.
  (b) The ring $R[t]$ is not a PID.

(2) Let A be an Artinian ring, and let M be an A-module which is annihilated by all nilpotent ideals in A. Prove that M is a semisimple A-module.

Proof. Since A is Artinian, $J(A)$ is nilpotent. Thus, M is a $A/J(A)$-module. Because $A/J(A)$ is semisimple, $\oplus_{m \in M} A \twoheadrightarrow M \to 0$ is a quotient of semisimple modules, which is semisimple. □

(3) (a) Determine the radical of the ideal $I = (y^2 - 1, x^2 - (y+1)x + 1)$ in $\mathbb{C}[x, y]$. You may write your answer in any form which allows you to easily see if a given polynomial $f(x, y)$ is in the radical or not.
  (b) Determine if the inclusion $I \subseteq \sqrt{I}$ is an equality.

Proof. (a) Note that the radical of I is the intersection of all maximal ideals that contains I, which corresponds to the points in $V(I)$.
First, setting $y^2 - 1 = 1$ gives $y = \pm 1$. If $y = 1$, then $0 = x^2 - (y+1)x + 1 = (x-1)^2$. If $y = -1$, then $x^2 + 1 = 0$. Thus, $V(I) = \{(1, 1), (i, -1), (-i, -1)\}$. Thus,

$$\begin{aligned}\sqrt{I} &= (x - i, y + 1) \cap (x - 1, y - 1) \cap (x + i, y + 1) \\ &= \left(y^2 - 1, (x^2 + 1)(x - 1), (y - 1)(x^2 + 1), (x - 1)(y + 1)\right)\end{aligned}$$

(b) This is a strict inequality since no term in I contains a monomial of the form y but $(x-1)(y+1) = xy - y + x - 1 \in \sqrt{I}$ has.

□

(4) Classify all groups of order 50.

Proof. Since any index 2 subgroup is normal, G has a normal subgroup of order 25, which is the Sylow 5-subgroup P. Then $G \cong \mathbb{Z}/2 \ltimes_{\varphi} P$ where $\varphi : \mathbb{Z}/2 \to \mathrm{Aut}(P)$ is a group homomorphism.
  (i) Assume $P \cong \mathbb{Z}/25$. If φ is trivial, then $G \cong \mathbb{Z}/50$. Because $\mathrm{Aut}(\mathbb{Z}/25)$ is abelian, any embedding of $\mathbb{Z}/2$ gives the same semidirect product; hence,

$$G \cong \langle a, b \mid a^{25} = b^2 = 1, ab = ba^{24} \rangle$$

  (ii) Assume $P \cong (\mathbb{Z}/5\mathbb{Z})^2$.
Assume φ is nontrivial. Then the image of φ is an order 2 elements in $\mathrm{GL}(2, 5)$, which satisfies $x^2 - 1$. Thus, the minimal polynomial of φ is either $x - 1$, $x + 1$ or $x^2 - 1$. In the first case, φ is trivial. In the last case, φ sends $-1 \in \mathbb{Z}/2$ to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which gives $\mathbb{Z}/5 \times D_{10}$. In the second case, φ sends $-1 \in \mathbb{Z}/2\mathbb{Z}$ to $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$, which gives

$$G \cong \langle a, b, c \mid a^5 = b^5 = c^2 = 1, ab = ba, ac = ca^4, bc = cb^4 \rangle.$$

So there are 5 groups of order 50 up to isomorphism.

□

(5) Consider the polynomial $p(x) = x^{16} - \alpha x^{10} - \alpha x^6 + \alpha^2$, for α non-algebraic over $\mathbb{Q}$. Take $F = \mathbb{Q}(\alpha, \zeta)$, where $\zeta = e^{2\pi i/15}$.
  (a) Is $p(x)$ is irreducible over F?
  (b) Determine the Galois group $\mathrm{Gal}(K/F)$ for the splitting field K of $p(x)$ over F.

Proof. (a) We have $p(x) = x^{10}(x^6 - \alpha) - \alpha(x^6 - \alpha) = (x^{10} - \alpha)(x^6 - \alpha)$ is not irreducible over F.

(b) Note that roots of $x^6 - \alpha$ are of the form $\sqrt[2]{\zeta^5}^i \sqrt[6]{\alpha}$ with $1 \leq i \leq 6$ and roots of $x^{10} - \alpha$ are of the form $\sqrt[2]{\zeta^3}^j \sqrt[10]{\alpha}$ with $1 \leq j \leq 10$. Let L be the splitting field of $x^6 - \alpha$ over F. Then $L = F(\sqrt[6]{\alpha}, \sqrt{\zeta^5})$, which is a degree $6 \times 2 = 12$ extension. Because $x^6 - \alpha$ is separable, L/F is Galois and $\mathrm{Gal}(L/F) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ where the generators are $\sqrt{\zeta^5} \mapsto -\sqrt{\zeta^5}$ and $\sqrt[6]{\alpha} \mapsto \sqrt[6]{\alpha}\sqrt{\zeta^5}$.

Next, $K = L(\sqrt[5]{\sqrt{\alpha}}, \sqrt{\zeta^3})$ is a degree 10 extension over L. Since $x^{10} - \alpha$ is separable, K/L is Galois and $\mathrm{Gal}(K/L) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ where the generators are $\sqrt{\zeta^3} \mapsto -\sqrt{\zeta^3}$ and $\sqrt[10]{\alpha} \mapsto \sqrt[10]{\alpha}\sqrt{\zeta^3}$.

So Galois group $\mathrm{Gal}(K/F) \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/30$. (to be double-checked) □

(6) For a given group G, let G' denote the subgroup generated by the commutators $[a, b] = a^{-1}b^{-1}ab$ in G. Explicitly, $G' = \langle a^{-1}b^{-1}ab : a, b \in G \rangle \subseteq G$.

(a) Prove that G' is normal in G, and that G/G' is abelian.

(b) Show that if H is normal in G and G/H is abelian, then H contains G'.

(c) Prove that $S_5' = A_5$.

Proof. (1) For any $g \in G$ and $h \in G'$, we have

$$g^{-1}hg = hh^{-1}g^{-1}hg = h[h, g] \in G'.$$

Hence, $G' \triangleleft G$. Suppose $aG', bG' \in G/G'$. Then we have

$$(aG')(bG')(a^{-1}G')(b^{-1}G') = [a, b]G' = G' \Leftrightarrow abG' = baG'.$$

(2) Since G' is generated by all elements of the form $[a, b] = a^{-1}b^{-1}ab$, it suffices to show that $[a, b] \in H$ for any $a, b \in G$. Since G/H is abelian, we have

$$[a, b]H = aba^{-1}b^{-1}H = (aH)(bH)(a^{-1}H)(b^{-1}H) = H \Leftrightarrow [a, b] \in H.$$

Hence, $G' \leq H$.

(3) Notice that A_5 is the kernel of the sign map sgn : $S_5 \to \mathbb{Z}/2\mathbb{Z}$. For any $\sigma, \tau \in S_5$, we have $\mathrm{sgn}(\sigma\tau\sigma^{-1}\tau^{-1}) = 1$ since $\mathbb{Z}/2\mathbb{Z}$ is abelian; hence, $[S_5, S_5] \leq A_5$. Since $[S_5, S_5]$ is normal in S_5, $[S_5, S_5]$ is normal in A_5. Since A_5 is simple, $[S_5, S_5] = A_5$ or $[S_5, S_5] = 1$. However, we have

$$(12)(23)(12)(23) = (321) \neq 1.$$

Hence, $[S_5, S_5] = A_5$. □