ALGEBRA QUALIFYING EXAM FALL 2021

(1) Classify all groups of order $13^2 \times 7$.

Proof. Assume G is a group of order $13^2 \times 7$. Since 7 is the smallest prime divides $|G|$, G has a normal subgroup of index 7, namely the Sylow 13-subgroup P_{13}. Let $\mathbb{Z}/7 \cong K \leq G$ be a subgroup of order 7. Then G is a semidirect product of P_{13} and K.

(i) Assume $P_{13} \cong \mathbb{Z}/169$. Then $|\operatorname{Aut}(P_{13})| = 169 - 13 = 13 \times 12$ is coprime to 7; hence, any morphism from K to $\operatorname{Aut}(\mathbb{Z}/7)$ is trivial. Therefore $G \cong \mathbb{Z}/169 \times \mathbb{Z}/7$.

(ii) Assume $P_{13} \cong (\mathbb{Z}/13)^2$. If G is abelian then $G \cong (\mathbb{Z}/13)^2 \times \mathbb{Z}/7$.
Otherwise, assume G is nonabelian. Then $\operatorname{Aut}(P_{13}) \cong \mathrm{GL}(2, 13)$, which has order $(169 - 1)(169 - 13) = 7 \times 24 \times 13 \times 12$. Since any subgroup of order 7 in $\mathrm{GL}(2, 13)$ is a Sylow 7-subgroup and they are conjugate to each other, any embedding of $K \hookrightarrow \mathrm{GL}(2, 13)$ gives a nontrivial semidirect structure of $G \cong (\mathbb{Z}/13)^2 \rtimes \mathbb{Z}/7$.

□

(2) (a) If G is a transitive subgroup of S_n with $n > 1$ which has no fixed points, prove that there exists $g \neq 1$ in G so that g has no fixed points.

(b) Give an example of $G < S_n$ so that every element of G has fixed points but G itself has no fixed points.

Proof. (a) See Spring 2020 Q2 (a).

(b) Consider the subgroup $G = \langle (123), (12)(45) \rangle < S_5$. It is easy to checked that G has no fixed point and every element of G admits at least a fixed point.

□

(3) Let b be any integer coprime to 7 and consider the polynomial $f_b(x) = x^3 - 21x + 35b$. Show that f_b is irreducible over $\mathbb{Q}$. Write P for the set of $b \in \mathbb{Z}$ such that b is coprime to 7 and the Galois group of f_b is the alternating group. Find P. (Hint: the discriminant of the cubic polynomial of the form $x^3 + px + q$ is given by $-4p^3 - 27q^2$.)

Proof. Since b is coprime to 7, f_b is irreducible by Eisentein's criterion (applying to 7).

Let's compute the discriminant

$$D = \Delta^2 = 4 * 21^3 - 27 * 35^2 b^2 = 3^3 * 7^2 (28 - 25b^2).$$

Because $\operatorname{Gal}(f_b) \leq A_3$ if and only if $\Delta \in \mathbb{Q}$, which only happens when $b = 1$. Thus, we must have $b = 1$. Since f_b is irreducible, we must have $3 \mid |\operatorname{Gal}(f_b)|$. Because $|A_3| = 3$, we have $A_3 \cong \operatorname{Gal}(f_b)$. □

(4) Suppose R is a commutative local ring so R has a unique maximal ideal $\mathfrak{m}$. Show if $x \in \mathfrak{m}$, then $1 - x$ is invertible. Show if, in addition, R is Noetherian and $\mathfrak{a} \subset R$ an ideal such that $\mathfrak{a}^2 = \mathfrak{a}$ then $\mathfrak{a} = 0$.

Proof. Suppose $x \in \mathfrak{m}$. Since $1 \notin \mathfrak{m}$, we must have $1 - x \notin \mathfrak{m}$. Because R is local, we have $(1 - x) = R$ which means $1 - x$ must be invertible.

Assume R is Noetherian. Note $\mathfrak{a}$ is finite because R is Noetherian. Because $\mathfrak{a} \in \mathfrak{m}$ and R is local, we can apply Nakayama's lemma to $\mathfrak{a}$, which says $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}$ implies $\mathfrak{a} = 0$. □

(5) Let A be a finite dimensional central division algebra over a field F. Prove $[A, A] \neq A$.

Proof. See Fall 2018 Q6 (b). □

(6) Let $K \subset F$ be a nontrivial finite field extension. Prove $F \otimes_K F$ is not a domain.

Proof. We prove by contradiction. Assume that $F \otimes_K F$ is a domain. Therefore any subring of $F \otimes_K F$ is also a domain.

Let $\alpha \in F \setminus K$ and $f(x) = \operatorname{minipoly}_K(\alpha) \in K[x]$. Then $K(\alpha) \cong K[x]/(f)$ contains all roots of f since f is the minimal polynomial of its roots. Then we have $K(\alpha) \otimes_K K(\alpha)$ is a subring of

$F \otimes_K F$ and

$$K(\alpha) \otimes_K K(\alpha) \cong K(\alpha) \otimes_K K[x]/(f) \cong K(\alpha)[x]/(f).$$

Note that $x - \alpha \in K(\alpha)[x]/(f)$ and $(x - \alpha) \prod_{\alpha'} (x - \alpha') = f = 0 \in K(\alpha)[x]/(f)$ where α' ranges over all roots of $\frac{f(x)}{x-\alpha}$. It contradicts to the assumption that $K(\alpha) \otimes_K K(\alpha)$ is a domain. Thus, $F \otimes_K F$ is not a domain. $\square$