ALGEBRA QUALIFYING EXAM FALL 2020

(1) Let V_A be a $\mathbb{Q}[x]$-module that corresponds to a matrix $A \in \text{Mat}_n(\mathbb{Q})$. In other words, $V_A = \mathbb{Q}^n$, and the $\mathbb{Q}[x]$-module structure is defined by $f \cdot v = f(A)v$ for $f \in \mathbb{Q}[x]$, $v \in \mathbb{Q}^n$. The module V_A is called cyclic if there is $v \in V$ so that $\mathbb{Q}[x] \cdot v = V_A$. The matrix A is called cyclic if V_A is cyclic.

(a) Prove: V_A is cyclic if and only if $\text{Ann}(V_A)$ is generated by the characteristic polynomial $\chi(A)$.

(b) For any matrix A there is a cyclic matrix C such that $AC - CA = 0$.

Proof. (a) Since $\mathbb{Q}$ is a field, $\mathbb{Q}[x]$ is a PID. Since V_A is a finite dimensional $\mathbb{Q}$-module, V_A is a finitely generated torsion $\mathbb{Q}[x]$-module. By the structure theorem for finitely generated modules over a PID, we have $V_A \cong \mathbb{Q}[x]/(f_1) \oplus \cdots \oplus \mathbb{Q}[x]/(f_k)$ for some $f_i \in \mathbb{Q}[x]$ such that $f_1 \mid f_2 \mid \cdots \mid f_k$ are monic polynomials. Note that $\chi_A = f_1 \cdots f_k$ and the annihilator of V_A is generated by the minimal polynomial f_k. Therefore, V_A is cyclic if and only if $k = 1$, in which case $\text{Ann}(V_A) = (f_1)$ is generated by the characteristic polynomial.

(b) No idea.

□

(2) A finite dimensional algebra over a field has finitely many up to isomorphism simple left modules.

Proof. Let A be a finite dimensional algebra over a field k. Note that any simple R-module is of the form $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. The claim follows that A is Artinian, which implies A has finitely many maximal ideals. □

(3) Find all pairwise non-isomorphic groups of order 147 that contain no elements of order 49.

Proof. Let G be a group of order $147 = 3 \cdot 49$. By Sylow's theorems, G has a normal Sylow 7-subgroup P of order 49. Since G has no element of order 49, $P_7 \cong (\mathbb{Z}/7)^2$.

If G is abelian, then $G \cong P_7 \times P_3 \cong (\mathbb{Z}/7)^2 \times \mathbb{Z}/3$.

Assume that G is non-abelian. Since $\text{Aut}(P_7) \cong \text{GL}_2(\mathbb{Z}/7)$ is of order $7 * 32 * 9$. Because Sylow subgroups are conjugate to each other, it suffices to find one Sylow 3-subgroup Q in $\text{GL}_2(\mathbb{Z}/7)$. For example, $Q = \langle \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \rangle$ is a Sylow 3-subgroup of $\text{GL}_2(\mathbb{Z}/7)$. So G has the following presentation

(1) $\langle a, b, c \mid a^7 = b^7 = c^3 = 1,\ ab = ba,\ ac = ca^2,\ bc = cb \rangle$, which is given by $\mathbb{Z}/3 \to \text{GL}_2(\mathbb{Z}/7)$ where $c \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

(2) $\langle a, b, c \mid a^7 = b^7 = c^3 = 1,\ ab = ba,\ ac = ca^2,\ bc = cb^4 \rangle$, which is given by $\mathbb{Z}/3 \to \text{GL}_2(\mathbb{Z}/7)$ where $c \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$.

(3) $\langle a, b, c \mid a^7 = b^7 = c^3 = 1,\ ab = ba,\ ac = ca^2,\ bc = cb^2 \rangle$, which is given by $\mathbb{Z}/3 \to \text{GL}_2(\mathbb{Z}/7)$ where $c \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

□

(4) Prove that if for $f, g \in \mathbb{C}[x, y]$ the system of equations $f = g = 0$ has finitely many solutions, then the algebra $\mathbb{C}[x, y]/(f, g)$ is finite dimensional.

Proof. Since $\mathbb{C}[x, y] = \mathbb{C}[x][y]$ is a UFD, we can talk about gcd of two polynomials. Let $d = \gcd(f, g)$. If $d \notin \mathbb{C}^\times$, then d is a non-constant polynomial, which has infinitely many solutions in $\mathbb{C}^2$ and contradicts to the assumption. Therefore, we can assume that $1 = \gcd(f, g)$.

View f, g as elements in $\mathbb{C}(x)[y]$. Then $af + bg = 1$ for some $a, b \in \mathbb{C}(x)[y]$. Let $h \in \mathbb{C}(x)$ be the least common multiple of the denominators of a and b. Then $h(af + bg) = h \in (f, g)$ is a polynomial in $\mathbb{C}[x]$. Similarly, we have $s \in (f, g)$ where s is a polynomial in $\mathbb{C}[y]$. Thus, we have a $\mathbb{C}$-algebra epimorphism $\mathbb{C}[x, y]/(r, s) \twoheadrightarrow \mathbb{C}[x, y]/(f, g)$ and

$$\dim_{\mathbb{C}} \mathbb{C}[x, y]/(f, g) \leq \dim_{\mathbb{C}} \mathbb{C}[x, y]/(r, s) = \deg(r) \cdot \deg(s) < \infty.$$

□

(5) Show that a finitely generated projective module over a principal ideal domain is free.

Proof. See **??**. □

(6) Show that $p(x) = x^5 - 4x + 2$ is irreducible over $\mathbb{Q}$, and that it has exactly three real roots. Use this to show that the Galois group of $p(x)$ is S_5. Answer the question: Is $p(x) = 0$ solvable by radicals?

Proof. By Eisentein's criterion, $p(x)$ is irreducible over $\mathbb{Q}$. It has three real roots by Intermediate Value Theorem and its derivative $p'(x) = 5x^4 - 4$ has only two real roots. Thus, complex conjugation exchange two complex roots, which is a transposition in $\operatorname{Gal}(p(x))$. Because $p(x)$ is irreducible, we have $5 \mid [\mathbb{Q}[x]/p(x) : \mathbb{Q}] = |\operatorname{Gal}(p(x))|$; hence, $\operatorname{Gal}(p(x)) \subseteq S_5$ contains an element of order 5 by Cauchy's theorem. Therefore, $\operatorname{Gal}(p(x))$ has a 5-cycle. Since a 5-cycle and a transposition generates S_5, $\operatorname{Gal}(p(x)) \cong S_5$.

Because S_5 is not solvable, $p(x) = 0$ is not solvable by radicals. □