ALGEBRA QUALIFYING EXAM - FALL 2019

(1) Suppose p and q are primes with $p < q$. If $n \geq 0$ is an integer, then show that any finite group G of order pq^n is solvable.

Proof. Because $p < q$, p is the smallest prime divides the order of G; hence, G has a normal Sylow q-subgroup Q of order q^n. So we have a subnomral series $1 \triangleleft Q \triangleleft G$ with $G/Q \cong \mathbb{Z}/p$ is abelian. Therefore, to prove G is solvable, it suffices to show that Q is solvable. Since any p-group is nilpotent and any nilpotent group is solvable, we have Q is solvable. Hence, G is solvable. □

(2) Let G be a finite group, $H \leq G$ a subgroup and S a Sylow p-subgroup of G.
  (a) Show that the intersection of H with some conjugate of S is a Sylow p-subgroup of H.
  (b) Give an example to show that $H \cap S$ need not be a Sylow p-subgroup of H.

Proof. (a) Assume P is a Sylow p-subgroup of H. Then P is contained in some Sylow p-subgroup of G. Because Sylow subgroups are conjugate to each other, we have $P \subset gSg^{-1}$ for some $g \in G$. Therefore, $H \cap gSg^{-1} \supseteq P$. Since P is maximal, we have $H \cap gSg^{-1} = P$.
(b) Let $G = S_3$. $H = \langle(1,2)\rangle$ and $S = \langle(2,3)\rangle$. Then $H \cap S$ is trivial.

□

(3) Give an example of a field extension of degree 4 that has no intermediate subfield of degree 2 (hint: consider a Galois extension with group S_4).

Proof. Claim: For each n, there is a Galois extension whose Galois group is S_n.
Let $K/\mathbb{Q}$ be a field extension of Galois group S_4. Then S_3 is a maximal subgroup in S_4 with degree 4. The field extension $K^{S_3}/\mathbb{Q}$ is of degree 4 and has no intermediate field by Galois correspondence. □

(4) Prove that the subset $\{(u^3, u^2v, uv^2, v^3) \mid u, v \in \mathbb{C}\} \subset \mathbb{C}^4$ is algebraic.

Proof. Let $f(x) = x_1x_3 - x_2^2$ and $g(x) = x_3^2 - x_2x_4$. We claim that $V(f, g)$ is the desired set $S = \{(u^3, u^2v, uv^2, v^3) \mid u, v \in \mathbb{C}\}$.
Plug in the coordinates, it is easy to see $S \subset V(f, g)$. Thus, it suffices to show the other direction. □

(5) Let k be a field.
  (a) Prove that if $A, B \in M_n(k)$ are 3×3 matrices, then A and B are similar if and only if they have the characteristic and minimal polynomials.
  (b) Show the statement in the preceding point may fail for 4×4 matrices.

Proof. (a) The statement is obviously true in $\implies$ direction.
For the converse, we note that two matrices are similar if and only if they have same rational canonical form, which means they have the same invariant factors. If the minimal polynomial is of degree 2, then the other invariant factor is the quotient of the characteristic polynomial by the minimal polynomial, which must be the same. If the minimal polynomial is of degree 3, then two matrices have the same companion block of size 3×3. If the minimal polynomial is of degree 1, then two matrices are diagonalizable and they have the same eigenvalues of multiplicity 3.
(b) Let $k = \mathbb{C}$. Then $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ are not similar; however, they have same minimal polynomial $(x-1)^2$ and characteristic polynomial $(x-1)^4$.

□

(6) If R is a left Noetherian ring, then show that every element $a \in A$ that admits a left inverse actually admits a 2-sided inverse.

Proof. Consider the isomorphism $A^{op} \to \mathrm{End}_A(A)$ and show that any surjective endomorphism of Noetherian ring is an isomorphism. $\square$