Algebra Exam September 2014

Show your work. Be as clear as possible. Do all problems.
Hand in solutions in numerical order.

1. Let G be a group of order 56 having at least 7 elements of order 7. Let S be a Sylow 2-subgroup of G.

   (a) Prove that S is normal in G and $S = C_G(S)$.

   (b) Describe the possible structures of G up to isomorphism. (Hint: How does an element of order 7 act on the elements of S?)

orbit stab. formula →

2. Show that a finite ring with no nonzero nilpotent elements is commutative.

artin wedd

3. If $R = M_n(\mathbb{Z})$, and A is an additive subgroup of R, show that as additive subgroups $[R : A]$ is finite if and only if $R \otimes_{\mathbb{Z}} \mathbb{Q} = A \otimes_{\mathbb{Z}} \mathbb{Q}$.

details

4. Let R be a commutative ring with 1, n a positive integer and $A_1, \dots A_k \in M_n(R)$. Show that there is a noetherian subring S of R containing 1 with all the $A_i \in M_n(S)$.

Hilbert Basis thm

5. Let $R = \mathbb{C}[x, y]$. Show that there exists a positive integer m such that $((x + y)(x^2 + y^4 - 2))^m$ is in the ideal $(x^3 + y^2, y^3 + xy)$.

null stellensatz

6. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree $n \geq 5$. Let L be the splitting field of f and let $\alpha \in L$ be a zero of f. Given that $[L : Q] = n!$, prove that $\mathbb{Q}[\alpha^4] = \mathbb{Q}[\alpha]$.

use prev question

$1 \neq A_n \lhd S_n$

$A_n \leftrightarrow \mathrm{Gal}(L/\mathbb{Q}(\beta))$

Fall 2014

① $|G| = 56$ having atleast 7 elmts of order 7

S - sylow 2 subgroup of G.

(a) Prove S is normal in G & $S = C_G(S)$

(b) Describe all possible structures of G up to iso.

Pf (a) $|G| = 7 \cdot 8 = 7 \cdot 2^3$

now $n_7 \equiv 1 \mod 7$ & $n_7 = 1, \cancel{2}, \cancel{4}, 8$ if $n_7 = 1 \Rightarrow \exists$ only 6 elmts of order 7 $\Rightarrow \Leftarrow$.

so $\boxed{n_7 = 8}$

if $n_2 \neq 1$, then $n_2 = 7$

so that there are $7(8-1) = 7(7) = 49$ nontrivial elmts of order 2^i

But since $n_7 = 8 \Rightarrow 8(7-1) = 86 = 48$ elmts of order 7

$\Rightarrow 56 - 48 - 1 = 7$ - elmts left (nontrivial) so $n_2 \neq 7$

so $n_2 = 1 \Rightarrow |S| = 8$ & $S \triangleleft G$.

...

Now, if P is a sylow 7 subgrp, consider the action P on S.

$\ni$ $a: P \times S \to S$ then by the orbit stabilizer formula
$(p, s) \mapsto psp^{-1}$

$7 = |P| = |O_s||Stab_s|$ for any $s \in S$.

so then $|O_s| = 1$ or 7. If $|O_s| = 7$ $\Rightarrow$ for any $s, \tilde{s} \in S$ $\exists p \ni s = p\tilde{s}p^{-1}$
$\Rightarrow$ ord(s) = ord$(\tilde{s})$ $\Rightarrow$ since S must contain an element of order 2 (p-group) $\to S = \mathbb{Z}/_2 \times \mathbb{Z}/_2 \times \mathbb{Z}/_2$

if $|O_s| = 1 \Rightarrow s = psp^{-1} \Rightarrow sp = ps$ $\forall s, \forall p$
so $G \cong P \times S$ $\Rightarrow \Leftarrow$ since P is not normal.

so $S = \mathbb{Z}/_2 \times \mathbb{Z}/_2 \times \mathbb{Z}/_2$

Hence S is abelian & thus $S \subseteq C_G(s) = \{g \in G \mid gsg^{-1} = s \ \forall s \in S\}$

Now clearly if $C_G(s) \neq S$ then $C_G(s) = G$.

But then $G = P \times S \Rightarrow \Leftarrow$. so $C_G(s) = S$.

(b) since $S = \mathbb{Z}/_2 \times \mathbb{Z}/_2 \times \mathbb{Z}/_2$ & $P = \mathbb{Z}/_7$

then $G \cong \cancel{P \times S}$ or $G = P \rtimes S$ where

cannot be since $P \not\trianglelefteq G$

$\varphi : P \to \mathrm{Aut}(S) = GL_3(\mathbb{Z}/2)$ $(P:7)$

$\mathrm{Aut}(S)$: $0 \mapsto 0'$, a $b \mapsto b'$, b $c \mapsto c'$, c

$(p^3 - p^1)(p^3 - p)(p^3 - 1)$
$(\cdot 4)(\cdot 6)(7)$
$24 \cdot 7$
$8 \cdot 3$
$n_7 \equiv 1 \bmod 7$
$n_7 = \cancel{1}, 2, 4, 3, 8$
$12, 6, 24$

② R - finite ring w/ $\mathrm{Nil}(R) = 0 \Rightarrow R$ is comm.

Pf R - finite $\Rightarrow$ R is artinian $\Rightarrow J(R)$ is nilpotent

$\mathrm{Nil}(R) = 0 \Rightarrow R$ is Jacobson semisimple + artinian

$\Rightarrow R$ is semisimple $\Rightarrow$ apply artin wedderburn.

$\Rightarrow R = M_{n_1}(\Delta_1) \times \dots \times M_{n_k}(\Delta_k)$

but $n_i = 1 \ \forall i$ since $\mathrm{Ni}(R) = 0$ & R is finite $\Rightarrow \Delta_i$ is finite

$\Rightarrow$ field by little wedderburn.

$\Rightarrow R = F_1 \times \dots \times F_k \leftarrow$ commutative

4) R - commutative ring w/ 1. $n > 0$

$A_1, \ldots, A_k \in M_n(R)$

Show $\exists$ noetherian subring S of R $\ni$ $1 \in S$ & all the

$A_i \in M_n(S)$

~~Pf~~ Since R is commutative over some field k, then let $S = \langle \{ a_{ij}^p \} \rangle$ be the subring gen. by all entries of A_p $p \in \{1, \ldots k\}$. Then S is fin. gen. over a field $\Rightarrow$ S is noetherian by Hilbert basis theorem. (since $\mathbb{F}[x_1, \ldots, x_{kn^2}] \twoheadrightarrow S$) so then $A_p \in M_n(S)$ $\forall p$.

(3) $R = M_n(\mathbb{Z})$, A - additive subgroup of R. Show $[R : A]$ is finite iff $R \otimes_{\mathbb{Z}} \mathbb{Q} = A \otimes_{\mathbb{Z}} \mathbb{Q}$.

Since $R = M_n(\mathbb{Z})$, we can consider $R = \begin{bmatrix} \mathbb{Z} & \cdots & \mathbb{Z} \\ \vdots & & \\ \mathbb{Z} & & \mathbb{Z} \end{bmatrix}$ w/

$A = \begin{bmatrix} m_1\mathbb{Z} & \cdots & m_n\mathbb{Z} \\ & & \\ & & m_{n^2}\mathbb{Z} \end{bmatrix}$. Then $[R : A] < \infty$ iff $[\mathbb{Z} : m_i\mathbb{Z}] < \infty$ $\forall i$

iff $m_i \geq 1$ $\forall i$ iff $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = m_i\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$ iff $R \otimes_{\mathbb{Z}} \mathbb{Q} \simeq A \otimes_{\mathbb{Z}} \mathbb{Q}$.

(5) $R = \mathbb{C}[x,y]$. Show $(x+y)\cdot(x^2+y^4-2) \in \sqrt{I}$

$I = \langle x^3+y^2,\ y^3+xy \rangle$

Pf Consider $J = \langle (x+y)(x^2+y^2\cdot 4) \rangle = \langle p(x) \rangle$

Then $p(x) \in \sqrt{I} = Id(var(I))$ - nullstellensatz

iff $\langle p(x) \rangle = J \subseteq \sqrt{I}$

iff $var(J) \supseteq var(I)$

so if $x^3+y^2=0$ & $y^3+xy=0$

$\Rightarrow$ $-x^3 = (-x)^3 = y^2$ $\qquad y^3 = -xy \Rightarrow y=0$

$\downarrow y \neq 0$

$y^2 = -x$

if $y=0 \Rightarrow x=0$

if $y=-x \Rightarrow -x^3 = (-x)^2 = x^2$

$\Rightarrow x=0$ or $x \neq 0 \Rightarrow -x=1 \Rightarrow x=-1,\ y=1$

$\Downarrow$

$y=0$

so

$Var(I) = \{(0,0), (-1,1)\}$.

Now $p(0,0)=0$ & $p(-1,1) = 2(1+1-2) = 0$. so $p(x)^m \in I$.

Sep, Fall 2014

(6) $f(x) \in \mathbb{Q}[x]$ irred of deg $n \geq 5$. L - split field & let $\alpha \in L$, zero of f

If $[L:\mathbb{Q}] = n!$ prove $\mathbb{Q}[\alpha^4] = \mathbb{Q}[\alpha]$.

Pf Recall: $\exists$ no subgroups of S_n $\ni$ $3 \leq [S^n : H] < n$ for $n \geq 5$.

Since α has min polyn. of degree $n \Rightarrow [\mathbb{Q}(\alpha):\mathbb{Q}] = n$ & $\mathbb{Q}(\alpha^4) \leq \mathbb{Q}(\alpha)$.

Now by galois correspondence deg $\mathbb{Q}(\alpha^4)/\mathbb{Q}$ = index of some $S \leq Gal(L/\mathbb{Q}) = S^n$

(Clearly since f has degree n & $Gal(L/\mathbb{Q}) \hookrightarrow S^n$ then $Gal(L/\mathbb{Q}) = S^n$).

So then, deg $\mathbb{Q}(\alpha^4)/\mathbb{Q}$ must be 1, 2 or n by the lemma above.

Clearly if degree = n $\Rightarrow$ $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^4)$ since $[\mathbb{Q}(\alpha):\mathbb{Q}(\alpha^4)][\mathbb{Q}(\alpha^4):\mathbb{Q}] = [\mathbb{Q}(\alpha):\mathbb{Q}] = n$

~~Then~~ if degree = 1 then $\alpha^4 \in \mathbb{Q}$, but then $x^4 - \alpha^4$ is the minimal polyn. of α over $\mathbb{Q}$ $\Rightarrow \Leftarrow$ since $n \geq 5$.

~~Hence~~ So lastly if degree = 2 then $\alpha^4 = a + c\sqrt{b}$ for some $a, c, b \in \mathbb{Q}$

$\Rightarrow \alpha = \sqrt[4]{a + c\sqrt{b}}$ ← solvable by radicals.

But $n \geq 5$ so f is not solvable by radicals $\Rightarrow \Leftarrow$.

Thus deg = n $\Rightarrow \mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^4)$.