Fall 2013: Algebra Graduate Exam

Problem 1.

Let H be a subgroup of the symmetric group S_5. Can the order of H be 15, 20 or 30?

Proof. First note that the only normal subgroup of S_5 is A_5, which has order 60.

Case 1. Assume $|H| = 15$. Then H must have both a Sylow 5-subgroup and a Sylow 3-subgroup, and thus H must contain a 5-cycle and a 3-cycle. Since neither of the subgroups generated by these elements is normal in S_5, their product is not a subgroup. Therefore any subgroup of S_5 containing a 5-cycle and a 3-cycle has more than 15 elements.

Case 2. Assume $|H| = 20$. Of course, H must have both a Sylow 5-subgroup and a Sylow 2-subgroup, so

Case 2a. Assume the Sylow 2-subgroup contains a transposition. Then a 5-cycle and a transposition generates S_5, so $|H| = 120$, a contradiction.

Case 2b. If the Sylow 2-subgroup does not contain a transposition, it must contain element of the form $(s_1 s_2)(s_3 s_4)$. (...?)

Case 3. Assume $|H| = 30$. Thus H has a Sylow 5-subgroup, a Sylow 3-subgroup, and a Sylow 2-subgroup. Based on Case 2a, if H has a Sylow 2-subgroup, it must be of the form $(s_1 s_2)(s_3 s_4)$ (...?)

□

Problem 2.

Let R be a PID and M a finitely generated torsion module of R. Show that M is a cyclic R-module if and only if for any prime $\mathfrak{p}$ of R, either $\mathfrak{p}M = M$ or $M/\mathfrak{p}M$ is a cyclic R-module.

Proof. □

Problem 3.

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and suppose I is a proper non-zero ideal of R. The coefficients of a matrix $A \in M_n(R)$ are polynomials in $x_1, \ldots, x_n$ and can be evaluated at $\beta \in \mathbb{C}^n$; write $A(\beta) \in M_n(\mathbb{C})$ for the matrix so obtained. If for some $A \in M_n(R)$ and all $\alpha \in \text{Var}(I)$, $A(\alpha) = 0_{n\times n}$, show that for some integer m, $A^m \in M_n(I)$.

Proof. □

Problem 4.

If R is a noetherian unital ring, show that the power series ring $R[[x]]$ is also a noetherian unital ring.

Proof. □

Problem 5.

Let p be a prime. Prove that $f(x) = x^p - x - 1$ is irreducible over $\mathbb{Z}/p\mathbb{Z}$. What is the Galois group?

Hint. Observe that if α is a root of $f(x)$, then so is $\alpha + i)$ for $i \in \mathbb{Z}/p\mathbb{Z}$.

Proof. □

Problem 6.

Let $K \subset \mathbb{C}$ be the field obtained by adjoining all roots of unity in $\mathbb{C}$ to $\mathbb{Q}$. Suppose $p_1 < p_2$ are primes, $a \in \mathbb{C} \setminus K$, and write L for a splitting field of

$$g(x) = (x^{p_1} - a)(x^{p_2} - a)$$

over K. Assuming each factor of $g(x)$ is irreducible, determine the order and the structure of $\mathrm{Gal}(L/K)$.

Proof. □