Algebra Qualifying Exam - Fall 2013

1. Let H be a subgroup of the symmetric group S_5. Can the order of H be 15, 20 or 30?

2. Let R be a PID and M a finitely generated torsion module of R. Show that M is a cyclic R-module if and only if for any prime $\mathfrak{p}$ of R either $\mathfrak{p}M = M$ or $M/\mathfrak{p}M$ is a cyclic R-module.

3. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and suppose I is a proper non-zero ideal of R. The coefficients of a matrix $A \in M_n(R)$ are polynomials in $x_1, \ldots, x_n$ and can be evaluated at $\beta \in \mathbb{C}^n$; write $A(\beta) \in M_n(\mathbb{C})$ for the matrix so obtained. If for some $A \in M_n(R)$ and all $\alpha \in Var(I)$, $A(\alpha) = 0_{n\times n}$, show that for some integer m, $A^m \in M_n(I)$.

4. If R is a noetherian unital ring, show that the power series ring $R[[x]]$ is also a noetherian unital ring.

5. Let p be a prime. Prove that $f(x) = x^p - x - 1$ is irreducible over $\mathbb{Z}/p\mathbb{Z}$. What is the Galois group? (Hint: observe that if α is a root of $f(x)$, then so is $\alpha + i$ for $i \in \mathbb{Z}/p\mathbb{Z}$.)

6. Let R be a finite ring with no nilpotent elements. Show that R is a direct product of fields.

7. Let $K \subset \mathbb{C}$ be the field obtained by adjoining all roots of unity in $\mathbb{C}$ to $\mathbb{Q}$. Suppose $p_1 < p_2$ are primes, $a \in \mathbb{C} \setminus K$, and write L for a splitting field of

$$g(x) = (x^{p_1} - a)(x^{p_2} - a)$$

over K. Assuming each factor of $g(x)$ is irreducible, determine the order and the structure of $Gal(L/K)$.

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① $H \leq S_5$, can H have order 15, 20, 30.

if $|H| = 20, 30, 15 \Rightarrow$ $[S_5, H] = 6$ or $[S_5, H] = 4$ or $[S_5, H] = 8$

However by #5 on spring 2014 $[S_5, H] = 1, 2$ or 5.

Hence these cannot occur.

② R - PID, M - fig. torsion module. M is a cyclic R module iff for any prime $p \in R$ either $pM = M$ or M/pM is cyclic.

$(\Rightarrow)$ If M is acyclic R module $\Rightarrow M \simeq R/\text{ann}(M) \simeq Rx$

where $\text{ann}(M) = \{r \in R ; rx = 0\}$

But also $M \simeq M_{(p)} \leftarrow$ p-primary components; $M_{(p)} = \{m \in M ; p^n m = 0 \text{ some } n\}$

so then for any $q \neq p$ $qM = M$ since $q^n m \neq 0$

if $q = p$ then M/pM will be cyclic since M itself is cyclic.

$(\Leftarrow)$ if $pM = M$ or M/pM is cyclic for any given $p \in R$ then

Given $M \simeq M_{(p_1)} \oplus \cdots \oplus M_{(p_n)}$, then for $q \neq p_i$ obs $qM = M$ so suppose $q = p_i$

$M/p_i M = M_{(p_1)} \oplus \cdots \oplus M_{p_i}/p_i M \oplus \cdots N_{(p_n)}$ being cyclic

$\Rightarrow M/p_i M \simeq M_{(p_i)}/p_i M \Rightarrow M_{(p_u)} = 0 \ \forall \ u \neq i \Rightarrow M \simeq M_{(p_i)}$

so M is cyclic

(3) $R = \mathbb{C}[x_1, \dots x_n]$, I nonzero ideal in R.

If $A \in M_n(R)$ & $\forall \alpha \in \mathrm{Var}(I)$; $A(\alpha) = 0$, show $\exists m \geq A^m \in M_n(I)$

~~If~~ Since $R = \mathbb{C}[x \dots x_n]$ & $\mathbb{C}$ is alg. closed, R is noeth by Hilbert Basis theorem.

Now $A = (a_{ij})$ w/ $a_{ij} \in R$, then $a_{ij}(\beta) = 0$ $\forall \beta \in \mathrm{Var}(I)$

$\Rightarrow \mathrm{var}(I) \subseteq \bigcap_{ij} \mathrm{var}(a_{ij}) \Rightarrow \mathrm{Id}\,\mathrm{var}(I) \supseteq \bigcup_{ij} \mathrm{Id}(a_{ij}) = \bigcup_{ij} (a_{ij}) \ni a_{ij}$

so by Null stellensatz $\sqrt{I} \ni a_{ij}$ $\forall ij$ $\Rightarrow$ $\exists m_{ij} \geq a_{ij}^{m_{ij}} \in I$.

Now, R is noeth, so $\sqrt{I}$ is fin generated, $\sqrt{I} = \langle f_1, \dots, f_n \rangle$

w/ $f_i^{m_i} \in I$ some m_i. Let $M = \max\{m_i\}_{i=1}^n$, so $f_i^M \in I$ $\forall i$.

Now $a_{ij} \in \sqrt{I} \Rightarrow$ $a_{ij} = \sum_{k=1}^{n} c_{ij}^k f_k$, so $a_{ij}^s = \left(\sum^n c_{ij}^k f_k\right)^s$

$= \sum_{l=1}^{n \cdot s} B_l f_1^{\ell_1} \cdots f_n^{\ell_n} \geq \sum \ell_i = s$, so if $f_i^M \in I$, let $S = M \cdot k$

then atleast one $f_i^{k_i} \in I$ So $a_{ij}^s \in I$. Thus if let $m = n^2 \cdot M \cdot k$

then $A^m \in M_n(I)$.

(4) R - noetherian $\Rightarrow R[x]$ is noetherian.

Same as hilbert basis theorem proof but look at minimal coefficient (degree) instead of maximal.

(5) p - prime, Prime $f(x) = x^p - x - 1$ is irreducible over $\mathbb{Z}/p\mathbb{Z}$

What is the Galois gp?

If observe that if $f(\alpha) = 0 \Rightarrow \alpha^p - \alpha - 1 = 0$

then $f(\alpha+n) = (\alpha+n)^p - \alpha - n - 1 = \alpha^p + n^p - \alpha - n - 1 = n^p - n = 0$

since $n \in \mathbb{Z}/p$ so $n^p = n$.

Thus the roots of $f(x)$ are precisely $\{\alpha + n \mid n \in \mathbb{Z}/p\mathbb{Z}\}$.

Now consider the frobenius automorphism $\sigma: F_{p^i} \longrightarrow F_{p^i}$

$x \longmapsto x^p$

then $\sigma(\alpha) = \alpha^p = \alpha + 1$ so σ acts transitively on the roots of f & fixes $\mathbb{Z}/p\mathbb{Z}$. In particular the roots of f are precisely

$\{\alpha, \sigma(\alpha), \sigma^2(\alpha), \cdots, \sigma^{p-1}(\alpha)\}$, so by irreducibility of f we obtain $\mathrm{Gal}(f(x)) = \langle \sigma \rangle = \mathbb{Z}/p\mathbb{Z}$

6) R finite ring w/ no nilpotent elements. Show $R = \bigoplus F_i$ - fields

Pf R finite $\Rightarrow$ artinian $\Rightarrow$ $J(R)$ is nilpotent

$Nil(R) = 0$ + art $\Rightarrow$ $J(R) = 0$

$J(R) = 0$ + art $\Rightarrow$ R is semisimple $\Rightarrow$ can apply Artin Wedd.

So $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$

Since $Nil(R) = 0$ $\Rightarrow$ $n_i = 1$ $\forall i$

since R is finite $\Rightarrow$ D_i are finite $\Rightarrow$ D_i are fields by Little Wedderburn.

$\Rightarrow$ $R \cong F_1 \times \cdots \times F_k$.

(7) $K = \mathbb{Q}$ (roots of unity), let $\alpha \in \mathbb{C} \setminus K$, let L - splitting field of $g(x) = (x^{p_1} - a)(x^{p_2} - a)$ over K.

if $x^{p_i} - a$ is irred, determine order & structure of $Gal(L/K)$

Pf $x^{p_i} - a = 0$ ...

$x^{p_i} = a$

$x = \sqrt[p_i]{a}\,\omega$ ω - primitive p_ith root of unity.

So $L = K(\sqrt[p_1]{a}, \sqrt[p_2]{a})$, so then $K \overset{\sim}{-} K(\sqrt[p_1]{a}) \overset{\sim}{-} K(\sqrt[p_1]{a}, \sqrt[p_2]{a})$

so $[L : K] = p^2$, now let $\sigma : \sqrt[p_1]{a} \longmapsto \sqrt[p_1]{a}\,\omega$ $\tau : \sqrt[p_2]{a} \longmapsto \sqrt[p_2]{a}\,\eta$

then $|\langle\sigma\rangle| = p_1$, $|\langle\tau\rangle| = p_2$ and so $\langle\sigma, \tau\rangle$ generate L.

? $\sigma\tau\sigma^{-1} = \tau$, Hence the structure is