Fall 2012: Algebra Graduate Exam

Problem 1.

Use Sylow's theorems directly to find, up to isomorphism, all possible structures of groups of order $5 \cdot 7 \cdot 23$.

Proof. Sylow's theorems tell us that any group G must have

$$\begin{aligned} r_5 &\text{ Sylow 5-subgroups,} \\ r_7 &\text{ Sylow 7-subgroups, and} \\ r_{23} &\text{ Sylow 23-subgroups} \end{aligned}$$

where r_5, r_7, and r_{23} divide $5 \cdot 7 \cdot 23$, and $r_p \equiv 1 \bmod p$.

$$r_p = 1, 5, 7, 5 \cdot 7, 23, 5 \cdot 23, 7 \cdot 23, \text{ or } 5 \cdot 7 \cdot 23$$

considering the restriction on modulus, $r_5 \in \{1, 7 \cdot 23\}$, $r_7 = 1$, and $r_{23} = 1$. Let P and Q be the unique Sylow 23-subgroup and Sylow 7-subgroup respectively. Since $P \cap Q = 1$, $PQ \cong P \times Q$. Let R be a Sylow 5-subgroup.

Since $R \trianglelefteq G$ (why?), and R has a complement $P \times Q$, G is a semidirect product of R by $P \times Q$, that is $G = R \ltimes (P \times Q)$.

By Rotman Lemma 7.21, there is a homomorphism

$$\theta \colon \underbrace{R \to \mathrm{Aut}(P \times Q)}_{\mathbb{Z}_5 \to \mathbb{Z}_{22} \times \mathbb{Z}_6}.$$

But since $\gcd(5, 22) = \gcd(5, 6) = 1$, the only homomorphism is trivial. Therefore there is only one group of order $5 \cdot 7 \cdot 23$, the abelian group

$$G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{23}.$$

□

Problem 2.

Let A, B, and C be finitely generated $F[x] = R$ modules for F a field with C torsion free. Show that $A \otimes_R C \cong B \otimes_R C$ implies that $A \cong B$. Show by example that this conclusion can fail when C is not torsion free.

Proof. (From Nicolle)
R is a PID since F is a field, so by the *structure theorem for finitely generated modules over a PID*,

$$\begin{aligned} A &\cong T(A) \oplus R^n \\ B &\cong T(B) \oplus R^m \\ C &\cong R^t, \end{aligned}$$

were $T(M)$ denotes the torsion submodule of M. Since $A \otimes_R C \cong B \otimes_R C$, it follows that

$$(T(A) \oplus R^n) \otimes_R R^t \cong (T(B) \oplus R^m) \otimes_R R^t$$
$$(T(A) \otimes_R R^t) \oplus (R^n \otimes_R R^t) \cong (T(B) \otimes_R R^t) \oplus (R^m \otimes_R R^t)$$

Thus the free part of $A \otimes_R C$ is isomorphic to the free part of $B \otimes_R C$:

$$R^n \otimes_R R^t \cong R^m \otimes_R R^t,$$

so $n = m$. Similarly, the torsion submodules of $A \otimes_R C$ and $B \otimes_R C$ are isomorphic:

$$(T(A) \otimes_R R^t) \cong (T(B) \otimes_R R^t),$$

so $T(A) = T(B)$. Therefore,

$$A \cong T(A) \oplus R^n \cong T(B) \oplus R^m \cong B,$$

as desired.

As a counterexample, consider $A = B \oplus \mathrm{Ann}(C)$. Then

$$A \otimes_R C \cong B \otimes_R C \oplus \underbrace{\mathrm{Ann}(C) \otimes_R C}_{0} \cong B \otimes_R C,$$

but $A \ncong B$. □

Problem 3.

Working in the polynomial ring $\mathbb{C}[x,y]$, show that some power of $f(x,y) = (x+y)(x^2+y^4-2)$ is in $I = (x^3+y^2, y^3+xy)$.

Note. This is identical to the Problem 5 in the 2014 fall exam.

Proof. It is sufficient to show that $f(x,y)$ vanishes on $\mathrm{Var}(I)$; by Hilbert's Nullstellensatz, this implies that $f(x,y)^m \in I$ for some $m \in \mathbb{N}$.

First note that $y^3 + xy = y(y^2+x)$ vanishes when $y = 0$ or $x = -y^2$.
Case 1. Assume $y = 0$. Then $x^3 + y^2$ vanishes at $(0,0)$.
Case 2. Assume $x = -y^2$. Substituting this yields $(-y^2)^3 + y^2 = y^2(-y^4+1)$, so the polynomial vanishes at $(0,0), (-1,1), (-1,-1), (1,i), (1,-i)$ Checking these:

$$\begin{aligned}
0^3 + 0^2 &= 0^3 + 0 \cdot 0 &&= 0 \\
(-1)^3 + 1^2 &= 1^3 + (-1) \cdot 1 &&= 0 \\
(-1)^3 + (-1)^2 &= (-1)^3 + (-1)(-1) &&= 0 \\
1^3 + i^2 &= i^3 + 1 \cdot i &&= 0 \\
1^3 + (-i)^2 &= (-i)^3 + 1(-i) &&= 0.
\end{aligned}$$

Now it is enough to check that $f(x,y)$ vanishes on $\mathrm{Var}(I) = \{(0,0), (-1,1), (-1,-1), (1,i), (1,-i)\}$:

$$\begin{aligned}
f(0,0) &= \underbrace{(0+0)}_{0}(0^2 + 0^4 - 2) = 0 \\
f(-1,1) &= \underbrace{(-1+1)}_{0}((-1)^2 + 1^4 - 2) \\
f(-1,-1) &= (-1 + (-1))\underbrace{((-1)^2 + (-1)^4 - 2)}_{0} \\
f(1,i) &= (1+i)\underbrace{(1^2 + i^4 - 2)}_{0} \\
f(1,-i) &= (1 + (-i))\underbrace{(1^2 + (-i)^4 - 2)}_{0}.
\end{aligned}$$

Thus by Hilbert's Nullstellensatz, since f vanishes on $\mathrm{Var}(I)$, a power of f is in I. □

Problem 4.

For integers $n, m > 1$, let $A \subseteq M_n(\mathbb{Z}_m)$ be a subring with the property that if $x \in A$ with $x^2 = 0$ then $x = 0$. Show that A is commutative. Is the converse true?

Proof. The idea here it to show that A is semisimple, and so by Artin-Wedderburn can be written as

$$A \cong M_{n_1}(\Delta_1) \times \ldots \times M_{n_m}(\Delta_m)$$

where Δ_i is a field because it is finite and $n_i = 1$.

The converse is false. Let A be the ring generated by a single element with $n = m = 2$:

$$A = \left\langle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

Then A is commutative, but $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ while $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. □

Problem 5.

Let F be the splitting field of $f(x) = x^6 - 2$ over $\mathbb{Q}$. Show that $\mathrm{Gal}(F/\mathbb{Q})$ is isomorphic to the dihedral group of order 12.

Proof. Firstly, $F = \mathbb{Q}[\sqrt[3]{2}, \omega]$ where ω is a sixth root of unity. Then

$$\begin{aligned} &[\mathbb{Q}[\sqrt[3]{2}] : \mathbb{Q}] = 6, \text{ and} \\ &[F : \mathbb{Q}[\sqrt[3]{2}]] = \varphi(6) = 2, \end{aligned}$$

so $[F : \mathbb{Q}] = [F : \mathbb{Q}[\sqrt[3]{2}]] \cdot [\mathbb{Q}[\sqrt[3]{2}] : \mathbb{Q}] = 12$ and $\mathrm{Gal}(F/\mathbb{Q}) = 12$. Now consider the automorphisms

$$\tau : \begin{cases} \omega \mapsto \overline{\omega} \\ \sqrt[3]{2} \mapsto \sqrt[3]{2} \end{cases} \quad \text{and} \quad \sigma : \begin{cases} \omega \mapsto \omega \\ \sqrt[3]{2} \mapsto \omega \sqrt[3]{2} \end{cases} .$$

Now τ is of order 2 and σ is of order 6, and the dihedral relation is satisfied:

$$\begin{aligned} \sigma\tau\sigma\tau(\omega) &= \sigma\tau\sigma(\overline{\omega}) &&= \sigma\tau(\overline{\omega}) &&= \sigma(\omega) &&= \omega \\ \sigma\tau\sigma\tau(\sqrt[3]{2}) &= \sigma\tau\sigma(\sqrt[3]{2}) &&= \sigma\tau(\omega\sqrt[3]{2}) &&= \sigma(\overline{\omega}\sqrt[3]{2}) &&= \underbrace{\overline{\omega}\omega}_{1}\sqrt[3]{2} = \sqrt[3]{2}. \end{aligned}$$

□

Problem 6.

Given that all groups of order 12 are solvable, show that any group of order $2^2 \cdot 3 \cdot 7^2$ is solvable.

Proof. Let r_p denote the number of Sylow p-subgroups of G. Sylows theorems state that r_p divides $2^2 \cdot 3 \cdot 7^2$, so

$$\begin{aligned} r_2 &\in \{1, 3, 7, 3 \cdot 7, 7^2, 3 \cdot 7^2\} \\ r_3 &\in \{1, 2, 2^2, 7, 2 \cdot 7, 2^2 \cdot 7, 7^2, 2 \cdot 7^2, 2^2 \cdot 7^2\} \\ r_7 &\in \{1, 2, 2^2, 3, 2 \cdot 3, 2^2 \cdot 3\} \end{aligned}$$

also $r_p \equiv 1 \bmod p$, so

$$\begin{aligned} r_2 &\in \{1, 3, 7, 3 \cdot 7, 7^2, 3 \cdot 7^2\} \\ r_3 &\in \{1, 2^2, 7, 2^2 \cdot 7, 7^2, 2^2 \cdot 7^2\} \\ r_7 &= 1 \end{aligned}$$

This means that there is a unique—and thus normal—Sylow 7-subgroup, call it $N \cong \mathbb{Z}_7$. Therefore $G \cong N \rtimes K$ where K is a subgroup of order 12.

Now a group is solvable if it has a normal series whose factor groups are cyclic of prime order. Since K is solvable, it has a normal series

$$K = K_0 \leq K_1 \leq K_2 \leq \ldots \leq K_n = 1.$$

where K_i/K_{i+1} is a cyclic group of prime order. Moreover, since N is normal, NK_{i+1} is a subgroup of NK_i. Thus

$$G = NK_0 \leq NK_1 \leq NK_2 \leq \ldots \leq \underbrace{NK_n}_{N} \leq 1$$

is a normal series of G where $NK_i/NK_{i+1} \cong K_i/K_{i+1}$ is a cyclic group of prime order for $i \in \{0, 1, ..., n-1\}$, and $N/1 \cong N \cong \mathbb{Z}_7$ is a cyclic group of prime order. Therefore G is solvable. □