Algebra Graduate Exam

Fall 2012

Work all the problems. Be as explicit as possible in your solutions, and justify your statements with specific reference to the results that you use. Partial credit will be given for partial solutions.

1. Use Sylow's theorems directly to find, up to isomorphism, all possible structures of groups of order $5 \cdot 7 \cdot 23$.

2. Let A, B, and C be finitely generated $F[x] = R$ modules, for F a field, with C torsion free. Show that $A \otimes_R C \cong B \otimes_R C$ implies that $A \cong B$. Show by example that this conclusion can fail when C is not torsion free.

3. Working in the polynomial ring $\boldsymbol{C}[x, y]$, show that some power of $(x + y)(x^2 + y^4 - 2)$ is in $(x^3 + y^2, y^3 + xy)$.

4. For integers $n, m > 1$, let $A \subseteq M_n(\mathbf{Z}_m)$ be a subring with the property that if $x \in A$ with $x^2 = 0$ then $x = 0$. Show that A is commutative. Is the converse true?

5. Let F be the splitting field of $f(x) = x^6 - 2$ over $\mathbf{Q}$. Show that $\mathrm{Gal}(F/\mathbf{Q})$ is isomorphic to the dihedral group of order 12.

6. Given that all groups of order 12 are solvable show that any group of order $2^2 \cdot 3 \cdot 7^2$ is solvable.

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① use sylow to find $|G| = 5 \cdot 7 \cdot 23$

Pf $n_{23} \equiv 1 \bmod 23$ & $n_{23} = 1, 5, 7, 35 \Rightarrow \exists P \trianglelefteq G \ |P| = 23$.

$n_7 \equiv 1 \bmod 7$ & $n_7 = 1, 23, 5 \cdot 23 \Rightarrow \exists Q \trianglelefteq G \ ; |Q| = 7$

so $\exists |PQ| = 23 \cdot 7 \Rightarrow PQ \trianglelefteq G$ & $PQ \simeq P \times Q$ (b/c $P \cap Q = \{1\}$)

so consider $\varphi : R \longrightarrow Aut(P \times Q)$ where R - sylow subgp.

Then $Aut(P \times Q) \simeq \mathbb{Z}_6 \times \mathbb{Z}_{22}$, but $(5,6) = 1$ & $(5,22) \neq 1$

so $\varphi \equiv 0 \Rightarrow G \simeq R \times P \times Q \simeq \mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_{23}$.

② A, B fin. gen. $F[x] = R$ modules, F - field, C torsion free.

Show, $A \otimes_R C \simeq B \otimes_R C \Rightarrow A \simeq B$.

Show this fails if C is not torsion free.

Pf $F[x]$ - PID since F is a field

So then by fund thm of f.g. modules over PID:

$A \simeq tA \oplus R^k$ $B \simeq tB \oplus R^m$ $C \simeq R^j$ since C is torsion free

so $A \otimes_R C \simeq B \otimes C \Rightarrow (tA \oplus R^k) \otimes (R^j) \simeq (tB \oplus R^m) \otimes R^j$

$\Rightarrow tA \otimes R^j \simeq tB \otimes R^j$ and $R^{k+j} \simeq R^{m+j} \Rightarrow m = k$

and $tA \simeq tB \Rightarrow A \simeq B$.

Consider $A = B \oplus Ann(C)$. Then $A \otimes_R C \simeq (B \otimes_R C) \oplus (Ann(C) \otimes_R C) \simeq B \otimes_R C$

but $A \not\simeq B$

(3) $\mathbb{C}[x,y]$, show $(x+y)(x^2+y^4-2) \in \sqrt{I}$

$I = \langle x^3+y^2, y^3+xy \rangle$

Pf $p(x) \in \sqrt{I}$ iff $Var(I) \subseteq Var(p(x))$ by nullstellensatz.

So $Var(I) \Rightarrow$ $x^3 = -y^2$ $\quad$ $y^3 = -xy$

$(-x)^3 = y^2$ $\quad$ $y \neq 0$ or $y = 0 \Rightarrow x = 0$

$\Downarrow$ $\quad$ $\Downarrow$

$-x^3 = -x$ $\Leftarrow$ $y^2 = -x$

$x \neq 0$ $\quad$ $x = 0$

$\Downarrow$ $\quad$ $\Downarrow$

$x^2 = 1$ $\quad$ $y = 0$

$\Downarrow$ $\quad$ $\downarrow$

$x = \pm 1$ $\quad$ $(0,0)$

$(1, \pm i)$ $\quad$ $(-1, \pm 1)$

So $Var(I) = \{(0,0), (1, \pm i), (-1, \pm 1)\}$.

Now $p(0,0) = 0$, $p(1, \pm i) = 0$ & $p(-1, \pm 1) = 0$

So $p(x) \in \sqrt{I}$.

(4) $n, m > 1$, $A \leq M_n(\mathbb{Z}_m)$ be a subring w/ property that if $x \in A$ w/ $x^2 = 0 \Rightarrow x = 0$. Show A is commutative.

Is converse true?

pf $\mathbb{Z}_m$ is finite ring, M_n is finite alg over finite ring.

$\Rightarrow M_n(\mathbb{Z}_m)$ is a finite ring $\Rightarrow M_n(\mathbb{Z}_m)$ is artinian.

$\Rightarrow A$ is artinian. $\Rightarrow J(A)$ is nilpotent

Now if $x \in A \ni x^n = 0$ for n-smallest such integer > 0. Then:

n even $\Rightarrow$ $x^n = x^{n/2} x^{n/2} = 0 \Rightarrow x^{n/2} = 0 \Rightarrow \in \frac{n}{2} < n$.

n odd $\Rightarrow$ $(x^n)x = x^{n+1} = x^{n+1/2} x^{n+1/2} = 0 \Rightarrow x^{\frac{n+1}{2}} = 0 \Rightarrow \in \frac{n+1}{2} < n$ for $n > 1$.

so $\text{Nil}(A) = 0 \Rightarrow J(A) = 0$ so A is art + Jac. semisimple

$\Rightarrow A$ is semisimple $\Rightarrow$ can apply artin Wedderburn.

so $A \cong M_{n_1}(\Delta_1) \times \cdots \times M_{n_k}(\Delta_k)$

A is finite $\Rightarrow \Delta_i = F_i$ - fields by little Wedderburn.

$\text{Nil}(A) = 0 \Rightarrow n_i = 1 \ \forall i$

so $A \cong F_1 \times \cdots \times F_n$ - commutative.

Now suppose $A \leq M_2(\mathbb{Z}_2)$ w/ $A = \left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle$.

Then $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$ but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = 0$. so the converse is false.

(5) F - splitting field of $f(x) = x^6 - 2$ over $\mathbb{Q}$.

Show $Gal(F/\mathbb{Q}) \cong D_{12} = \langle \sigma, \tau \mid \sigma^6 = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$.

Pf $x^6 = 2$

$x = \sqrt[6]{2}\,\omega \leftarrow$ prim 6th root. $\Rightarrow$

F } deg 6

$\mathbb{Q}(\omega)$

} deg $\varphi(6) = \varphi(2)\cdot\varphi(3) = 2$

$\mathbb{Q}$

so $|Gal(F/\mathbb{Q})| = 12$.

Now, w/ τ: $\omega \mapsto \bar{\omega}$, $\sqrt[6]{2} \mapsto \sqrt[6]{2}$ & σ: $\sqrt[6]{2} \mapsto \sqrt[6]{2}\,\omega$, $\omega \mapsto \omega$

Then $K\langle\tau\rangle| = 2$ & $|\langle\sigma\rangle| = 6$.

w/ $\tau\sigma\tau(\sqrt[6]{2}\,\omega) = \tau\sigma(\sqrt[6]{2}\,\bar{\omega}) = \tau(\sqrt[6]{2}\,\omega\cdot\bar{\omega}) = \sqrt[6]{2}\,\omega\cdot\bar{\omega}$

but $\bar{\omega} = \omega^5$ so $\Rightarrow \sqrt[6]{2} = \sigma^{-1}(\sqrt[6]{2}\,\omega)$.

Thus, $Gal(F/\mathbb{Q}) = D_{12}$

(6) If any grp of order 12 is solvable $\Rightarrow$ since $|G| = 2^2\cdot 3\cdot 7^2$ is solvable.

Pf Consider $n_7 = 1 \bmod 7$ & $n_7 = 1, 2, 4, 3, 12, 6$. $\Rightarrow n_7 = 1$

hence $P \trianglelefteq G$ w/ $|P| = 7^2 \Rightarrow |G/P| = 12$. & is solvable.

so then $\exists$ H_i $0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_n = G/P$ & so:

$0 \trianglelefteq P \trianglelefteq PH_1 \trianglelefteq PH_2 \trianglelefteq \dots \trianglelefteq PH_n = G$. (since P is also solvable! b/c p-grp.

since $PH_i/PH_{i-1} \cong (PH_i/P)/(PH_{i-1}/P) \cong (H_i/H_i\cap P)/(H_{i-1}/H_i\cap P) \cong (H_i/P)/(H_{i-1}/P) \cong H_i/H_{i-1}$

so each quotient is cyclic & of prime order $\Rightarrow G$ is solvable.