→try

ALGEBRA QUALIFYING EXAM
September, 2006

1. (a) Find the number of Sylow p-subgroups of the symmetric group S_p. Here p is a prime.

(b) Use (a) to prove that

$$(p-1)! + 1 \equiv 0 \bmod p.$$

2. Let G be a finite solvable group and H a minimal (non-trivial) normal subgroup. Show H is isomorphic to a direct sum of cyclic groups of order p, for some prime p. (Hint: First show that the commutator subgroup H' of H is $\{e\}$.)

3. Let $m_1, m_2, \dots, m_n$ be positive integers which are pairwise relatively prime (that is, $\gcd(m_i, m_j) = 1$ for all $i \neq j$).

(a) Show that $F := \mathbb{Q}(\sqrt{m_1} + \sqrt{m_2} + \cdots + \sqrt{m_n}) = \mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}, \dots, \sqrt{m_n})$. (Hint: induction.)

(b) Show that $F = \mathbb{Q}(\sqrt{m_1} + \sqrt{m_2} + \cdots + \sqrt{m_n})$ is Galois over $\mathbb{Q}$. What is its Galois group?

→ **4.** Let k be a field, let R be a commutative k-algebra , and let $S = M_n(R)$ be all $n \times n$-matrices with entries in R. Choose $A_1, \dots, A_m \in S$. Show that there exists a (left) Noetherian k-subalgebra S_0 of S which contains all of the matrices A_i. (Hint: Consider the subalgebra $R_0 \subset R$ generated by all entries of $A_1, \dots, A_m \in S$.)

5. Let $R = \mathbb{C}[x, y, z]$, let $I = (x^2z^3 - y^2z + xyz - x^2y)$ be an ideal of R, and define $S = R/I$.

(a) Prove that the polynomial $x^2z^3 - y^2z + xyz - x^2y$ is irreducible in R. (Hint: consider it as an element of $\mathbb{C}[x, y][z]$.)

(b) Show that S is a Noetherian integral domain.

(c) Prove that in S the intersection of all maximal ideals is $\{0\}$.

→ **6.** Let k be a finite field and let R be a finite dimensional semi-simple k-algebra such that for all $r \in R$, there exists a positive integer $n = n(r) > 0$ such that $r^{n(r)}$ is in the center of R. Prove that R is commutative.

7. Let M be a finitely generated $\mathbb{Z}$-module with torsion submodule $T(M)$.

(a) Justify: $M/T(M)$ is a free $\mathbb{Z}$-module.

For parts (b) and (c), set $r(M) := \operatorname{rank}(M/T(M))$.

(b) Show that $r(M) = \dim_{\mathbb{Q}}(M \otimes_{\mathbb{Z}} \mathbb{Q})$.

(c) Assume that

$$0 \to M \to N \to P \to 0$$

→ tensor with Q

is an exact sequence of $\mathbb{Z}$-modules. Show that

→ keeps exact

$$r(N) = r(M) + r(P).$$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{n\cdot(n-1)\cdots(n-r+1)}{r!}$$

~~$(s_1 \cdots s_r)$~~

$I = \bigcap$ max ideals.

$Var(I) = \bigcup Var(\text{max ideals})$

$= Var(\bigcap \text{max ideals})$

$I = \sqrt{I} = \sqrt{\bigcap \text{max ideals}}$ ~~$= \bigcap$ max ideals~~

$\mathbb{C}(x,y,z)$

$\downarrow$

$I = \sqrt{I} = Id(Var(I))$

$= Id(\bigcup \alpha_i) = \bigcap_i (X - \alpha_i)$

$= \sqrt{\bigcap Id(\alpha_i)}$

Algebra : Fall 2006

① (a) Find the # of Sylow p-subgroups of S_p for p prime.

$|S_p| = p!$, hence the order has only one factor of p, hence a Sylow p-subgroup will be order p. An order p subgroup is generated by a p-cycle.

We know there are $\frac{p(p-1)\cdots(p-p+1)}{p} = \frac{p!}{p} = (p-1)!$ p-cycles in S_p.

Each Sylow p-sg. will have $p-1$ p-cycles & the identity, hence there are $\frac{(p-1)!}{(p-1)} = \underline{(p-2)!}$ Sylow p-subgroups.

(b) Prove that $(p-1)! \equiv -1 \pmod p$

We have $r_p = (p-2)!$ and by Sylow's theorems, $r_p \equiv 1 \pmod p$.
On the other hand, $(p-1) \equiv -1 \pmod p$, hence: $r_p(p-1) = (p-1)!$

$(p-1)! = r_p(p-1) = (1)\cdot(-1) = (-1) \pmod p$.

② Let $|G| < \infty$, solvable, $H \trianglelefteq G$ minimal normal.
Show H is a direct sum of cyclic sg's order p for some prime p.

Consider $[H,H] = H' \trianglelefteq H$; now we want to show $H' \trianglelefteq G$ as well.
Choose $xyx^{-1}y^{-1} \in H'$ and $g \in G$; then:

$$gxyx^{-1}y^{-1}g^{-1} = \underbrace{gxg^{-1}}_{\substack{a \\ \in H}}\underbrace{gyg^{-1}}_{\substack{b \\ \in H}}\underbrace{gx^{-1}g^{-1}}_{\substack{a^{-1} \\ \in H}}\underbrace{gy^{-1}g^{-1}}_{\substack{b^{-1} \\ \in H}} = aba^{-1}b^{-1} \in [H,H]$$

$\leftarrow$ since $H \trianglelefteq G$

Therefore $H' \trianglelefteq G$, hence by minimality of H, $H' = H$ or $H' = \{e\}$.

Case $H' = H$: Then, since $H' = H$, we have $H^{(n)} = H$ for all n, hence derived sg. will not terminate, hence not solvable; contradiction since $H \trianglelefteq G$ $\Rightarrow$ H solvable since G is solvable.

Case $H' = \{e\}$: Then H is abelian & finite, hence $H \cong \underbrace{A_{p_1} \oplus \cdots \oplus A_{p_k}}_{p_i\text{-primary parts}}$; $k=1$ since

H has no nontrivial

Now, since $H \cong A_{p_1} \oplus \cdots \oplus A_{p_k}$, we have subgroups $A_{p_i} \leq H$ such that, for $g \in G$, $gA_{p_1}g^{-1} = A_{p_1}$ since $gHg^{-1} = H$ & conjugation preserves orders.

Therefore $k=1$.

So $H \cong A_p \cong \mathbb{Z}_{p^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_\ell}}$. Then H has subgroup $\overbrace{\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}^{\ell} \leq H$, such that, for $g \in G$, $g(\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p)g^{-1} = g\mathbb{Z}_p g^{-1} \oplus \cdots \oplus g\mathbb{Z}_p g^{-1} = \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$ since $gHg^{-1} = H$ & conj. preserves order.

Therefore $e_i = 1$ $\forall i$, and $\underline{H \cong \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}$.

③ $m_1, \ldots, m_n$ positive, pairwise coprime

(a) Show $F = \mathbb{Q}(\sqrt{m_1} + \sqrt{m_2} + \cdots + \sqrt{m_n}) = \mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}, \ldots, \sqrt{m_n})$

Consider tower:

$$\begin{tikzcd}
\mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}, \ldots, \sqrt{m_n}) \arrow[d, dash, "2"] \\
\mathbb{Q}(\sqrt{m_1}, \ldots, \sqrt{m_{n-1}}) \arrow[d, dash, "2"] \\
\vdots \arrow[d, dash, "2"] \\
\mathbb{Q}(\sqrt{m_1}) \arrow[d, dash, "2"] \\
\mathbb{Q}
\end{tikzcd}$$

→ pairwise coprime, so no way to get one from product of the others, so $\mathbb{Q}(\sqrt{m_1}, \ldots, \sqrt{m_{n-1}}) \underset{\neq}{\subseteq} \mathbb{Q}(\sqrt{m_1}, \ldots, \sqrt{m_n})$

Therefore we get each extension is degree 2

So $[\mathbb{Q}(\sqrt{m_1}, \ldots, \sqrt{m_n}) : \mathbb{Q}] = 2^n$

There is an order 2 elt for each extension w/ $\sigma_i(\sqrt{m_i}) = -\sqrt{m_i}$

So $\mathrm{Gal}(\mathbb{Q}(\sqrt{m_1}, \ldots, \sqrt{m_n})/\mathbb{Q}) \cong (\mathbb{Z}_2)^n$

Now see that if $\sqrt{m_1} + \cdots + \sqrt{m_n} = \sigma(\sqrt{m_1} + \cdots + \sqrt{m_n}) = \sigma(\sqrt{m_1}) + \cdots + \sigma(\sqrt{m_n})$ then σ fixes each $\sqrt{m_i}$, hence $\sigma = id$. So $\sum \sqrt{m_i}$ is ~~nothing~~ fixed by id, hence only in the fixed field field of id, i.e. only in $\mathbb{Q}(\sqrt{m_1}, \ldots, \sqrt{m_n})$.

Now consider the tower

$$\begin{tikzcd}
 & \mathbb{Q}(\sqrt{m_1}, \ldots, \sqrt{m_n}) \arrow[d, dash, "2"] \arrow[dl, dash] \\
\mathbb{Q}(\sqrt{m_1} + \cdots + \sqrt{m_n}) \arrow[r, dash] & \mathbb{Q}(\sqrt{m_1}, \ldots, \sqrt{m_{n-1}}) \arrow[d, dash, "2^{n-1}"] \\
 & \mathbb{Q}
\end{tikzcd}$$

So $[\mathbb{Q}(\sqrt{m_1}, \ldots, \sqrt{m_n}) : \mathbb{Q}(\sqrt{m_1} + \cdots + \sqrt{m_n})] = 1$

So $\underline{\mathbb{Q}(\sqrt{m_1}, \ldots, \sqrt{m_n}) = \mathbb{Q}(\sqrt{m_1} + \cdots + \sqrt{m_n})}$

③ $m_1, m_2, \dots, m_n$ positive integers, pairwise coprime.

(a) Show $\mathbb{Q}(\sqrt{m_1} + \cdots + \sqrt{m_n}) = \mathbb{Q}(\sqrt{m_1}, \dots, \sqrt{m_n})$

$\subseteq$ clear, so must only show $\boxed{\supseteq}$

Try for $n=2$: $\underset{\substack{\shortparallel \\ K}}{\mathbb{Q}(\sqrt{m} + \sqrt{n})} \subseteq \underset{\substack{\shortparallel \\ L}}{\mathbb{Q}(\sqrt{m}, \sqrt{n})}$.

$$(\sqrt{m} + \sqrt{n})^2 = m + n + 2\sqrt{nm} \in K$$

$$\rightarrow \sqrt{nm} \in K.$$

now $\underset{\substack{\cap \\ K}}{(\sqrt{m} + \sqrt{n})}(\sqrt{nm}) = \underbrace{\sqrt{n}\,m + n\sqrt{m}}_{\in K}.$

$$\text{so } \underbrace{\sqrt{n}\,m + n\sqrt{m}}_{\in K} - \underbrace{m(\sqrt{m} + \sqrt{n})}_{\in K} = n\sqrt{m} - m\sqrt{m}$$

$$= (n-m)\sqrt{m} \in K$$

$$\Rightarrow \sqrt{m} \in K \Rightarrow K = L$$

$$\Rightarrow \sqrt{n} \in K.$$

now procede by induction...

Suppose $\mathbb{Q}(\sqrt{m_1} + \cdots + \sqrt{m_{n-1}}) = \mathbb{Q}(\sqrt{m_1}, \dots, \sqrt{m_{n-1}})$

Then $\mathbb{Q}(\sqrt{m_1}, \dots, \sqrt{m_{n-1}})(\sqrt{m_n}) = \mathbb{Q}(\sqrt{m_1} + \cdots + \sqrt{m_{n-1}})(\sqrt{m_n})$

$$= \mathbb{Q}(\sqrt{m_1} + \cdots + \sqrt{m_{n-1}}, \sqrt{m_n})$$

$$(\subseteq \mathbb{Q}(\sqrt{m_1} + \cdots + \sqrt{m_n}))$$

Let $\alpha = \sqrt{m_1} + \cdots + \sqrt{m_{n-1}}$

$\beta = \alpha + \sqrt{m_n}$

WTS: $\mathbb{Q}(\beta) \supseteq \mathbb{Q}(\alpha)(\sqrt{m_n})$ ($\subseteq$ trivial)

$\beta^2 = (\alpha + \sqrt{m_n}) = \alpha^2 + 2\sqrt{m_n}\,\alpha + m_n$

$\Rightarrow \sqrt{m_n} =$

④ k field, R comm k-algebra, $S = M_n(R)$ all $n \times n$ matrices with entries in R. Choose $A_1, \dots, A_m \in S$ & show $\exists$ noetherian k-subalgebra $S_0 \subseteq S$ containing all the A_i's.

R is a k-algebra, hence so is $M_n(R)$.

[Recall that a finitely-generated k-algebra R over noetherian ring k (true since k field) is itself noetherian:

Let $\{r_1, \dots, r_n\}$ be the generating set of R; then $\exists$ surjective hom. $f : k[r_1, \dots, r_n] \twoheadrightarrow R_0$; k noeth $\Rightarrow k[r_1, \dots, r_n]$ noeth by Hilb Basis

$\Rightarrow R_0 = f(k[r_1, \dots, r_n])$ noetherian since f is homomorphism.

Now recall that $\{$entries of A_i's$\} \subseteq R$, hence let

$R_0 = \langle$entries of A_i's$\rangle$ be the k-alg. generated by entries of the A_i's.

and then $M_n(R_0) \subseteq M_n(R)$ a k subalgebra; hence if $M_n(R_0)$ is finitely-generated then it is noetherian.

Consider the set $B = \left\{ M_{ij\ell} = \underset{j}{\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & r & \vdots \\ 0 & \cdots & 0 \end{pmatrix}} i \;\middle|\; \begin{array}{l} i, j = 1, \dots, n \\ \ell = 1, \dots, m \end{array}, \; \begin{array}{l} r \text{ the } ij^{\text{th}} \\ \text{entry of } A_\ell \end{array} \right\}$

Clearly $|B| < \infty$, but $\langle B \rangle = M_n(R_0)$, hence $M_n(R_0)$ is a finitely generated k-algebra, hence Noetherian.

---

⑤ $R = \mathbb{C}[x, y, z]$, $I = (x^2z^3 - y^2z + xyz - x^2y)$ and let $S = R/I$.

(a) Prove $x^2z^3 - y^2z + xyz - x^2y$ irreducible in R

Consider as an element of $\mathbb{C}[x,y][z]$; then $= x^2z^3 + y(x-y)z - x^2y$, hence irreducible with Eisenstein $p = y$, a prime in $\mathbb{C}[x,y]$.

(b) Show S is noeth integral domain:

I prime since the poly is irred, hence R/I int. domain. R is noeth by Hilbert Basis, so R/I noeth.

(c) The Jacobson radical of R/I is $\{0\}$

See that $I = \sqrt{I} = Id(Var(I)) = Id(\bigcup_i \{\alpha_i\}) = \sqrt{\bigcap_i Id(\{\alpha_i\})}$

I prime

$\overset{\circ}{=} \sqrt{\bigcap_i (X-\alpha_i)} = \bigcap_i \sqrt{(X-\alpha_i)} = \bigcap_i (X-\alpha_i)$

$a \in \sqrt{\bigcap (X-\alpha_i)}$

$\Rightarrow a^k \in \bigcap_i (X-\alpha_i) \Rightarrow a^k \in (X-\alpha_i)\ \forall i \Rightarrow a \in (X-\alpha_i)\ \forall i$ since ideals are prime

$\Rightarrow a \in \bigcap_i (X-\alpha_i)$

So $I = \bigcap_i (X-\alpha_i)$, hence an intersection of max ideals, hence $J(R) \subseteq I$.

Therefore, $J(R/I) \cong J(R)/I \cong 0$ since $J(R) \subseteq I$.

by ideal correspondence of R/I & R

---

⑥ k finite field, R fin-dim s.s. k-algebra s.t. $\forall r \in R\ \exists n = n(r) > 0$ such that $r^n \in Z(R)$. Prove R commutative

R s.s. $\Rightarrow R \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$ by Art-Wedd, and thus each division ring D_i must also be a fin-dim'l k-algebra, hence each is finite since k a finite field, hence the D_i's are fields by little Wedderburn.

$\Rightarrow R \cong M_{n_1}(F_1) \oplus \cdots \oplus M_{n_k}(F_k)$.

Now the centers of matrix rings are the scalar matrices, hence $Z(R) \cong F_1 \oplus \cdots \oplus F_k$; also, we have $\forall r \in R$, $\exists n$ s.t. $r^n \in Z(R)$, hence for $r \in R$, $r = (a_1, \ldots, a_k)$, $a_i \in M_{n_i}(F_i)$, we have n such that $r^n = (a_1^n, \ldots, a_k^n)$ and $a_i^n \in F_i\ \forall i$. $\Rightarrow$ a_i^n invertible or zero

If a_i^n is invertible: $a_i^n b = 1 \Rightarrow a_i a_i^{n-1} b = 1 \Rightarrow$ a_i invertible

If $a_i^n = 0$: a_i is nilpotent.

Hence all $a_i \in M_{n_i}(F_i)$ are nilpotent or invertible; for $n_i > 1$ there are elts that are neither, hence $n_i = 1\ \forall i$, hence $R \cong F_1 \oplus \cdots \oplus F_k$, hence R is commutative

⑦ M fin-gen $\mathbb{Z}$-module with torsion submodule $T(M)$

(a) $M/T(M)$ is free

By Fund Thm of Modules over PID, $M \cong \mathbb{Z}^k \oplus T(M)$, hence we have $M/T(M) \cong \mathbb{Z}^k \oplus T(M)/T(M) \cong \mathbb{Z}^k$ free.

(b) Let $rk(M) = rk(M/T(M))$ & show $rk(M) = \dim_{\mathbb{Q}}(M \otimes_{\mathbb{Z}} \mathbb{Q})$

$$M \otimes_{\mathbb{Z}} \mathbb{Q} \cong (\mathbb{Z}^k \oplus T(M)) \otimes \mathbb{Q} \cong (\mathbb{Z}^k \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus (T(M) \otimes_{\mathbb{Z}} \mathbb{Q})$$

$$\cong (\mathbb{Z} \otimes \mathbb{Q})^k \oplus (T(M) \otimes \mathbb{Q})$$

$$\overset{①}{\cong} (\mathbb{Q})^k \oplus 0 \cong \mathbb{Q}^k$$

① $S \cong S \otimes_R R$ for S an R-module

② $T(M)$ a finite abelian group; let be order n and then for $a \otimes q \in T(M) \otimes \mathbb{Q}$, $a \otimes q = a \otimes \frac{q}{n} n = na \otimes \frac{q}{n} = 0 \otimes \frac{q}{n} = 0$, hence $T(M) \otimes \mathbb{Q} = 0$

Hence $\dim_{\mathbb{Q}}(M \otimes \mathbb{Q}) = \dim_{\mathbb{Q}}(\mathbb{Q}^k) = k = rk(M/T(M)) := rk(M)$

(c) Assume $0 \to M \to N \to P \to 0$ exact seq of $\mathbb{Z}$-modules. Show that $rk(N) = rk(M) + rk(P)$.

$\mathbb{Q}$ is a flat module, so $0 \to M \otimes \mathbb{Q} \to N \otimes \mathbb{Q} \to P \otimes \mathbb{Q} \to 0$ is also exact, and is equal to:

$$0 \to \mathbb{Q}^{rk(M)} \to \mathbb{Q}^{rk(N)} \to \mathbb{Q}^{rk(P)} \to 0$$

Each is a free $\mathbb{Q}$-module, hence projective, hence sequence splits getting $\mathbb{Q}^{rk(N)} \cong \mathbb{Q}^{rk(M)} \oplus \mathbb{Q}^{rk(P)} \Rightarrow rk(N) = rk(M) + rk(P)$