ALGEBRA QUALIFYING EXAM FALL 2005

Work all the problems. Be as explicit as possible in your solutions and justify your statements with specific reference to the results that you use. Partial credit will be given for partial solutions.

1. Let G be a group with $|G| = p^n q^m$ for $p < q$ primes and assume that the order of $[p]_q$ in the multiplicative group $(\mathbf{Z}/q\mathbf{Z})^*$ is larger than n. Show that there are subgroups $\langle e \rangle \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_{n+m} = G$ with each $H_j \lhd H_{j+1}$ and H_{j+1}/H_j cyclic of prime order.

2. Let $F \subseteq L$ be finite fields with $[L : F] = 3$. If $\alpha \in F$ show that there is $\beta \in L$ satisfying $\beta^3 = \alpha$.

3. If $p(x) = x^8 + 6x^4 + 1 \in \mathbf{Q}[x]$ and if $\mathbf{Q} \subseteq M \subseteq \mathbf{C}$ is a splitting field for $p(x)$ over $\mathbf{Q}$, argue that $\mathrm{Gal}(M/\mathbf{Q})$ is solvable.

4. Let R be a commutative ring with 1 and let $x_1, \ldots, x_n \in R$ so that $x_1y_1 + \cdots + x_ny_n = 1$ for some $y_j \in R$. Let $A = \{(r_1, \ldots, r_n) \in R^n \mid x_1r_1 + \cdots + x_nr_n = 0\}$. Show that $R^n \cong_R A \oplus R$, that A has n generators as an R module, and that when $R = F[x]$ for F a field then A_R is free of rank $n - 1$.

5. Let $R = \mathbf{C}[x_1, \ldots, x_n]$ and let I be a nonzero proper ideal of R. If $A \in M_k(R)$ and $A(\alpha) = 0_{k\times k}$ for all $\alpha \in \mathrm{Var}(I)$, show that for some $s > 0$, $A^s \in M_k(I)$.

6. If R is a right Artinian algebra over $\mathbf{C}$, show there is an integer $m \geq 1$ so that if $x \in R$ and $x^k = 0$ for some integer $k \geq 1$, then $x^m = 0$: that is, the indices of nilpotence of the nil elements of R are bounded.