ALGEBRA EXAM SEPTEMBER 2004

Do as many problems as you can

1. Up to isomorphism describe all groups of order $399 = 3 \cdot 7 \cdot 19$. For each group find the order of its center and the order of its commutator subgroup.

2. Suppose R is a finite dimensional algebra over a field F with 1 and $U(R)$,the group of units in R, is abelian. Show that the Jacobson radical $J(R)$ and $R/J(R)$ are commutative.

3. Let L be a subfield of the finite field K of characteristic p. Let $\alpha \in K$ with minimal polynomial $v(x)$ of degree d over L. Show that $v(x)$ splits over K and that for some $q = p^m$ the roots of $v(x)$ in K are $\{\alpha, \alpha^q, \ldots, \alpha^{q^{d-1}}\}$.

4. Let R be a commutative ring with 1 and M, N, V all R-modules.

(a) If M and N are projective show that $M \otimes_R N$ is also a projective R-module.

(b) Let

$$\mathrm{Tr}(V) = \{\sum_{i=1}^{n} \phi_i(v_i) | \phi_i \in \mathrm{Hom}_R(V, R), v_i \in V, n = 1, 2, \ldots\}.$$

If $1 \in \mathrm{Tr}(V)$ show that up to isomorphism some finite direct sum V^k contains R as an R-module direct summand.

5. Show that any surjective ring homomorphism $f : R \to R$ of a left Noetherian ring R must be an isomorphism. Give an example to show this may be false if the ring is not noetherian.

6. In $\mathbb{C}[x, y]$ show that some power of $(x + y)(x^2 + y^4 - 2)$ is in the ideal $(x^3 + y^2, y^3 + xy)$.