510 A/B

Algebra Qualifying Exam September 2002

Partial credit is given for partial solutions

1. Let k be a field and let S_n act on the polynomial ring $k[X_1, \dots, X_n]$ by permuting the variables, i.e. $\sigma \cdot f = f^\sigma$ where

$$f^\sigma(X_1, \dots, X_n) = f(X_{\sigma(1)}, \dots, X_{\sigma(n)}).$$

Show that for any given f, the number of distinct polynomials of the form f^σ is a divisor of $n!$.

2. Show that there are exactly 2 groups of order $11 \cdot 43^2$.

3. Let $f(X) = X^3 - X - 1 \in \mathbb{Q}[X]$. Find the splitting field K of f over $\mathbb{Q}$, the Galois group of the extension $\mathbb{Q} \subseteq K$ and give the number of subfields of K of each degree.

4. Let k be an algebraically closed field, and let $R = k[X_1, \dots, X_n]$.
a) Show that if $\mathfrak{p}$ is any prime ideal of R, then $\mathfrak{p}$ is an intersection of maximal ideals.
b) If $n = 2$, show that the ideal $\mathfrak{p} = (X_1 + X_2)$ is prime and describe all maximal ideals $\mathfrak{m}$ such that $\mathfrak{p} \subseteq \mathfrak{m}$.

5. Let R be a commutative algebra over the field k, and $A_1, \dots A_t \in M_n(R)$ be $n \times n$-matrices with entries in R. Show that there exists a k-subalgebra $S \subseteq M_n(R)$ containing A_i, $1 \leq i \leq t$, such that S is (left) noetherian.
(hint: Try to find a subalgebra $R_0 \subseteq R$ such that $S = M_n(R_0)$.)

6. Let R be a finite ring. Show that if $x, y \in R$ satisfy $xy = 1$, then they also satisfy $yx = 1$.
(hint: First consider the case R is semi-simple.)