ALGEBRA QUALIFYING EXAM (MATH 510AB)

FALL 2000

(1) Describe all groups of order $3 \cdot 17 \cdot 23$ up to isomorphism.

(2) Let G be a finitely generated Abelian group so that every <u>proper</u> homomorphic image of G is cyclic. Prove that G is cyclic or that $|G| = p^2$ for p a prime.

(3) Let $K \subseteq \mathbb{C}$ be a splitting field over $\mathbb{Q}$ of $x^5 - 5$. Describe $\text{Gal}(K/\mathbb{Q})$. Describe those fields $\mathbb{Q} \subseteq M \subseteq K$ with M Galois over $\mathbb{Q}$, and for these find $\text{Gal}(M/\mathbb{Q})$.

(4) Let $\overline{F}$ be an algebraic closure of the field F. If $M \subseteq F[x_1, \ldots, x_n]$ is a maximal ideal, show that $V(M) = \{\alpha \in \overline{F} \times \cdots \times \overline{F} | f(\alpha) = 0 \text{ for all } f \in M\}$ in finite and not empty.

(5) Let $M \subseteq \mathbb{Q}$ be Noetherian $\mathbb{Z}$-submodule. For N a $\mathbb{Z}$-submodule of M, show M/N is finite (as a set) $\Leftrightarrow M \otimes_{\mathbb{Z}} \mathbb{Q} \cong N \otimes_{\mathbb{Z}} \mathbb{Q}$.

(6) If R is a right Artinian ring and $x^3 = x$ for all $x \in R$, show: R is commutative; R is finite; and R has $2^a 3^b$ elements for some $a, b \geq 0$.