Algebra Definitions

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1 Groups

1.1 Notation and definitions

1.1.1 Basic definitions

Definition 1.1.1.1 (Normal subgroup)**.** Let G be a group and K be a subgroup of G. If $gkg^{-1} \in K$ for all $k \in K$ and $g \in G$, then K is called a normal subgroup of G and is denoted $K \trianglelefteq G$.

Definition 1.1.1.2 (Simple group)**.** A group G is called a simple group is a group whose only normal subgroups are $\{e\}$ and G.

Definition 1.1.1.3 (Semidirect product)**.** Let $K \trianglelefteq G$ and $Q \leq G$. A group G is a semidirect product of K by Q (denoted $G = K \ltimes Q$) if there exists $Q_1 \cong Q$ such that Q_1 is a complement of K in G, that is $K \cap Q_1 = 1$ and $KQ_1 = G$.

1.1.2 Galois Theory

Definition 1.1.2.1 (Normal series)**.** A normal series of a group G is a sequence of subgroups

$$G = G_0 \geq G_1 \geq \ldots \geq G_n = 1$$

in which $G_{i+1} \trianglelefteq G_i$ for all i.

Definition 1.1.2.2 (Factor groups)**.** The factor groups of a normal series are the groups G_i/G_{i+1} for $i = 0, 1, \ldots, n-1$.

Definition 1.1.2.3 (Length)**.** The length of a a normal series is the number of nontrivial factor groups.

Definition 1.1.2.4 (Solvable group)**.** A finite group is solvable if it has a normal series whose factor groups are cyclic of prime order.

1.1.3 Centralizer/Normalizer

Definition 1.1.3.1 (Center)**.** The center of a group G, denoted by $Z(G)$, is the set of all $a \in G$ that commute with every element of G.

Definition 1.1.3.2 (Centralizer)**.** The centralizer of a subset S of a group G is defined to be

$$C_G(S) = \{g \in G \mid gs = sg \text{ for all } s \in S\}.$$

Definition 1.1.3.3 (Normalizer)**.** The centralizer of a subset S in the group G is defined to be

$$N_G(S) = \{g \in G \mid gS = Sg\}.$$

Definition 1.1.3.4 (Commutator)**.** If $a, b \in G$, the commutator of a and b, denoted $[a, b]$, is

$$[a, b] = aba^{-1}b^{-1},$$

and the commutator subgroup of G, denoted G', is the subgroup of G generated by all of the commutators.

Definition 1.1.3.5 (Class equation)**.** Partition G into its conjugacy classes, with x_i the representative of the ith conjugacy class. The class equation of the finite group G is

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)].$$

1.1.4 Group Actions

Definition 1.1.4.1 (Group action)**.** Let G be a group and X be a set. Then a group action on X is a function $\varphi\colon G \times X \to X$ denoted $\varphi(g, x) = g \cdot x$ and satisfying

(i) Identity: group action by the identity is trivial for all $x \in X$: $1 \cdot x = x$.

(ii) Compatibility: $(gh) \cdot x = g \cdot (h \cdot x)$.

And X is called a G-set.

Definition 1.1.4.2 (Orbit)**.** The orbit of an element $x \in X$ is denoted by

$$G \cdot x = \{g \cdot x \mid g \in G\}$$

Definition 1.1.4.3 (Stabilizer subgroup)**.** The stabilizer subgroup of G with respect to $x \in X$ is denoted

$$G_x = \{g \in G \mid g \cdot x = x\}$$

Definition 1.1.4.4 (Transitive)**.** A group action is called transitive is for each $x, y \in X$ there exists some $g \in G$ such that $g \cdot x = y$.

1.2 Theorems

Theorem 1.2.1 (First isomorphism theorem)**.** If $\varphi\colon G \to H$ is a group homomorphism then $\ker(\varphi) \trianglelefteq G$ and $G/\ker(\varphi) \cong \varphi(G)$.

Theorem 1.2.2 (Second isomorphism theorem)**.** Let G be a group with $S \leq G$ and $N \trianglelefteq G$. Then

1. $SN \leq G$
2. $S \cap N \trianglelefteq S$, and
3. $(SN)/N \cong S/(S \cap N)$.

Strictly speaking, N does not have to be a normal subgroup as long as S is a subgroup of the normalizer of N, $S \leq N_G(N)$.

Theorem 1.2.3 (Third isomorphism theorem)**.** Let G be a group with normal subgroup $N \trianglelefteq G$. Then

1. If $K \leq G$ (resp. $K \trianglelefteq G$) such that $N \subseteq K \subseteq G$, then $K/N \leq G/N$ (resp. $K/N \trianglelefteq G/N$).
2. Every subgroup (resp. normal subgroup) of G/N is of the form K/N, for some subgroup (resp. normal subgroup) $K \subset G$ such that $N \subseteq K \subseteq G$.
3. If $K \trianglelefteq G$ such that $N \subseteq K \subseteq G$, then $(G/N)/(K/N) \cong G/K$.

Theorem 1.2.4 (Simplicity of the A_n)**.** A_n is simple for all $n \geq 5$.

Theorem 1.2.5 (Sylow's theorem)**.**

(i) If P is a Sylow p-subgroup of a finite group G, then all Sylow p-subgroups of G are conjugate to P.

(ii) If there are r Sylow p-subgroups, then r divides $|G|$ and $r \equiv 1 \bmod p$.

Theorem 1.2.6 (Fundamental Theorem of Abelian Groups)**.** If G and H are finite abelian groups, then $G \cong H$ if and only if, for all primes p, they have the same elementary divisors.

Theorem 1.2.7. Let G be a finite group and p be the least prime divisor of $|G|$. Then if H is a subgroup of G such that $[G : H] = p$, then $H \trianglelefteq G$.

2 Fields

2.1 Notation and definitions

2.1.1 Basic definitions

Definition 2.1.1.1 (Degree of a field extension)**.** Suppose that E/k is a field extension. Then E may be considered as a vector space over k. The dimension of this vector space is called the degree of the field extension and is denoted by $[E : k]$.

Definition 2.1.1.2 (Field automorphism)**.** A field automorphism of a field K is an isomorphism $\phi\colon K \to K$. In particular,

$$\phi(a + b) = \phi(a) + \phi(b) \text{ and}$$
$$\phi(ab) = \phi(a)\phi(b).$$

Definition 2.1.1.3 (Splitting field)**.** A splitting field of a polynomial p over a field K is a field extension $L \supseteq K$ over which p factors into linear factors.

Definition 2.1.1.4 (Separable polynomial)**.** A polynomial p is called separable if if factors into distinct linear factors in its splitting field.

Definition 2.1.1.5 (Separable extension)**.** A separable extension is an field extension $E \supseteq F$ such that for every $\alpha \in E$, the minimal polynomial of α over F is a separable polynomial.

Definition 2.1.1.6 (Normal extension)**.** A normal extension $K \supseteq L$ is one for which every polynomial that is irreducible over K either has no root in L or splits into linear factors in L.

Definition 2.1.1.7 (Galois extension)**.** A Galois extension is an algebraic field extension E/F that is normal and separable.

Definition 2.1.1.8 (Galois group)**.** Let $E \supseteq F$ be a field extension. The Galois group $\mathrm{Gal}(E/F)$ is the set of automorphisms of E that fix F under function composition.

Definition 2.1.1.9 (Galois correspondence)**.** Let $E \supseteq F$ be a finite, Galois extension. The Galois correspondence is the bijection between intermediate fields $F \supseteq K \supset E$ and subgroups of the Galois group E/F.

Definition 2.1.1.10 (Trace)**.** ???

Definition 2.1.1.11 (Norm)**.** ???

Definition 2.1.1.12 (Radical extension)**.** A radical extension of a field K is an extension that is obtained by adjoining a sequence of nth roots of elements of K.

Definition 2.1.1.13 (Finite field)**.** A finite field is a field with a finite number of elements. Note: any finite field has p^k elements for some prime p and $k \in \mathbb{N}$.

Definition 2.1.1.14 (Cyclotomic extension)**.** A cyclotomic extension $\mathbb{Q}(\xi_n)$ of $\mathbb{Q}$ is an extension formed by adjoining a primitive nth root of unity.

Definition 2.1.1.15 (Algebraic closure)**.** An algebraic closure of a field K is an algebraic extension F/K such that F contains a root for every non-constant polynomial in $F[x]$.

2.2 Theorems

Theorem 2.2.1 (Isomorphism extension theorem)**.** Let F be a field and $\phi\colon F \to F'$ an isomorphism. Then if E is an extension field of F, ϕ can be extended into an isomorphism $\tau\colon E \to E'$.

Theorem 2.2.2 (Fundamental theorem of Galois theory)**.** Let E/k be a finite Galois extension with Galois group $G = \mathrm{Gal}(E/k)$. The function

$$\gamma\colon \mathrm{Sub}(\mathrm{Gal}(E/k)) \to \mathrm{Int}(E/k),$$

defined by $H \mapsto E^H$, is an order reversing bijection whose inverse maps $B \mapsto \mathrm{Gal}(E/B)$.

Theorem 2.2.3 (Primitive element theorem)**.** Finite separable extensions are simple.

3 Commutative Algebra

3.1 Notation and definitions

3.1.1 Basic definitions

Definition 3.1.1.1 (Maximal ideal)**.** An ideal $\mathfrak{m}$ of R is maximal if $\mathfrak{m} \subsetneq R$ and there is no ideal K of R such that $\mathfrak{m} \subsetneq K \subsetneq R$.

Definition 3.1.1.2 (Localization)**.** ???

Note. Localization is a formal way to introduce the "denominators" to a given ring or module.

Definition 3.1.1.3 (Integral element)**.** Let B be a ring and $A \subset B$ a subring. Then an element $b \in B$ is called integral over A if for some $n \geq 1$, there exist $a_j \in A$ such that $b^n = a_{n-1}b^{n-1} + \cdots + a_1 b + a_0$.

Definition 3.1.1.4 (Integral extension)**.** A ring B is called an integral extension of $A \subset B$ if every element of B is integral over A.

Note. The set of elements of B that are integral over A is called the integral closure of A in B.

Definition 3.1.1.5 (Unique factorization domain)**.** A unique factorization domain is an integral domain in which every non-zero non-unit element can be written as the product of prime elements uniquely.

Note. In general every prime is irreducible, but in a UFD, the converse is true.

Definition 3.1.1.6 (Principal ideal domain)**.** A principal ideal domain is one in which every ideal is generated by a single element. That is, if R is a ring and $I \subseteq R$ is an ideal of R, then $I = (a)$ for some element $a \in R$.

Definition 3.1.1.7 (Noetherian ring)**.** A Noetherian ring is a ring that satisfies the ascending chain condition on ideals, that is given any chain

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq I_{k+1} \subseteq \cdots$$

there exists some n after which $I_n = I_{n+1} = I_{n+2} = \cdots$.

Definition 3.1.1.8 (Variety)**.** Let k be an algebraically closed field, and F a subset of $k[x_1, \ldots, x_n]$. Then the variety defined by F is

$$\mathrm{Var}(F) = \{a \in k^n | f(a) = 0 \text{ for all } f \in F\}$$

Definition 3.1.1.9 (Zariski topology)**.** The Zariski topology is a topology on algebraic varieties where the closed sets on k^n are $\mathrm{Var}(F)$ for some $F \subset k[x_1, \ldots, x_n]$.

Note. For any two ideals of polynomials I and J,

1. $V(I) \cup V(J) = V(IJ) = \mathrm{Var}(I \cap J)$, and
2. $V(I) \cap V(J) = V(I + J)$.

3.2 Theorems

Theorem 3.2.1 (Eisenstein criterion)**.** Let D be an integral domain and $f(x) = a_n x^n + \ldots + a_1 x + a_0$ where $a_i \in D$, and so $f(x) \in D[x]$. Then if there exists a prime ideal $\mathfrak{p}$ of D such that

1. $a_i \in \mathfrak{p}$ for each $i < n$,
2. $a_n \notin \mathfrak{p}$, and
3. $a_0 \notin \mathfrak{p}^2$,

then f cannot be written as the product of two non-constant polynomials in $D[x]$.

Theorem 3.2.2 (Hilbert basis theorem)**.** A polynomial ring $R[x]$ over a Noetherian ring R is Noetherian.

Theorem 3.2.3 (Hilbert's Nullstellensatz)**.** *F12.3, S13.3.*
Let k be a field and $\overline{k}$ be an algebraically closed field extension. Consider the polynomial ring $k[x_1, \ldots, x_n]$, and let I be an ideal in this ring. Hilbert's Nullstellensatz states that if $p \in k[x_1, \ldots, x_n]$ vanishes on $\mathrm{Var}(I)$, then $p^r \in I$ for some $r \in \mathbb{N}$.

4 Modules

4.1 Notation and definitions

4.1.1 Basic definitions

Definition 4.1.1.1 (Irreducible module)**.** An irreducible module or a simple module over a ring R are nonzero modules whose only submodules are the module itself and the zero module.

Definition 4.1.1.2 (Torsion element)**.** An element m of a module M over a ring R is called a torsion element of the module if there exists (a non zero divisor) $r \in R$ such that $rm = 0$.

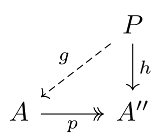
Definition 4.1.1.3 (Torsion module)**.** A module M over a ring R is called a torsion module if all of its elements are torsion elements.

Definition 4.1.1.4 (Free module)**.** A free module is a module that has a basis, E. That is

1. E is a generating set for M, and

2. E is linearly independent: $r_1e_1 + \ldots + r_ne_n = 0_M$ implies $r_1 = \ldots = r_n = 0_R$.

Definition 4.1.1.5 (Projective module)**.** A left R-module P is projective if whenever $p\colon A \to A''$ is a surjective module homomorphism and $h\colon P \to A''$ is any module homomorphism, then there exists a homomorphism $g\colon P \to A$ with $h = p \circ g$.



Note. A left R-module P is projective if and only if every short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} P \longrightarrow 0$$

is split, that is, $B \cong A \oplus P$.

Definition 4.1.1.6 (Modules over PIDs)**.** ???

Definition 4.1.1.7 (Chain conditions)**.** ???

Definition 4.1.1.8 (Tensor products)**.** Let A and B be modules over a commutative ring R. Then $A \otimes_R B$ is an R module where $\phi\colon A \times B \to A \otimes_R B$ defined by $(a, b) \overset{\phi}{\mapsto} a \otimes b$ is a middle linear map:

1. $\phi(a + a', b) = \phi(a, b) + \phi(a', b)$

2. $\phi(a, b + b') = \phi(a, b) + \phi(a, b')$

3. $\phi(ar, b) = \phi(a, rb)$

Where any bilinear map $h \colon A \times B \to Z$ can be written as $h = \widetilde{h} \circ \phi$ for some unique $\widetilde{h}$.

Note. The tensor product is the freest bilinear operation.

Definition 4.1.1.9 (Exact sequences)**.** An exact sequence is a sequence of objects and morphisms between them

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} G_n$$

such that $\operatorname{Im}(f_k) = \ker(f_{k+1})$.

4.2 Theorems

Theorem 4.2.1 (Structure theorem for finitely generated modules over a PID)**.** For every finitely generated module M over a PID R, there is a unique decreasing sequence of proper ideals

$$(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$$

such that M is isomorphic to the sum of cyclic modules:

$$M \cong \bigoplus_{i=1}^{n} R/(d_i) = R/(d_1) \oplus R/(d_2) \oplus \cdots \oplus R/(d_n).$$

Note. The number of d_is that are equal to zero is the dimension of the free part of M.

5 Noncommutative Rings

5.1 Notation and definitions

5.1.1 Basic definitions

Definition 5.1.1.1 (Artinian Rings)**.** A ring is called left artinian if it has descending chain condition on left ideals.

Definition 5.1.1.2 (Jacobson radical)**.** The Jacobson radical of R, denoted $J(R)$, is the intersection of all of the maximal left ideals in R.

Definition 5.1.1.3 (Jacobson semisimple)**.** A ring R is called Jacobson semisimple if $J(R) = (0)$.

Definition 5.1.1.4 (Division rings)**.** A division ring is a "possibly noncommutative field"; that is D is a ring in which $1 \neq 0$ and ever nonzero element $a \in D$ has a multiplicative inverse.

5.2 Theorems

Theorem 5.2.1 (Artin-Wedderburn theorem)**.** *Spring 2012, problem 3.*
Every semisimple ring R is a direct product,

$$R \cong \mathrm{Mat}_{n_1}(\Delta_1) \times \cdots \times \mathrm{Mat}_{n_m}(\Delta_m)$$

Theorem 5.2.2 (Skolem-Noether theorem)**.** Let A be a central simple k-algebra over a field k and let B and B' be isomorphic simple k-subalgebras of A. If $\psi \colon B \to B'$ is an isomorphism, then there exists a unit $u \in A$ with $\psi(b) = ubu^{-1}$ for all $b \in B$.

Theorem 5.2.3 (Wedderburn's theorem on finite division rings)**.** Every finite division ring D is a field.

Theorem 5.2.4 (Maschke's theorem)**.** *Spring 2012, problem 3.*
Let G be a finite group and K a field whose characteristic does not divide the order of G. Then $K[G]$, the group algebra of G, is semisimple.